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New results on multiplication in Sobolev spaces

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ABSTRACT

We consider the Sobolev (Bessel potential) spaces $H^\ell(\mathbf{R}^d, \mathbf{C})$, and their standard norms $\| \ \|_\ell$ (with ℓ integer or non-integer). We are interested in the unknown sharp constant $K_{\ell m n d}$ in the inequality $\|fg\|_\ell \leqslant K_{\ell m n d}\|f\|_m\|g\|_n$ ($f \in H^m(\mathbf{R}^d, \mathbf{C})$, $g \in H^n(\mathbf{R}^d, \mathbf{C})$; $0 \leqslant \ell \leqslant m \leqslant n$, $m + n - \ell > d/2$); we derive upper and lower bounds $K^{\pm}_{\ell m n d}$ for this constant. As examples, we give a table of these bounds for $d = 1$, $d = 3$ and many values of (ℓ, m, n); here the ratio $K^-_{\ell m n d}/K^+_{\ell m n d}$ ranges between 0.75 and 1 (being often near 0.90, or larger), a fact indicating that the bounds are close to the sharp constant. Finally, we discuss the asymptotic behavior of the upper and lower bounds for $K_{\ell, b\ell, c\ell, d}$ when $1 \leqslant b \leqslant c$ and $\ell \to +\infty$. As an example, from this analysis we obtain the $\ell \to +\infty$ limiting behavior of the sharp constant $K_{\ell, 2\ell, 2\ell, d}$; a second example concerns the $\ell \to +\infty$ limit for $K_{\ell, 2\ell, 3\ell, d}$. The present work generalizes our previous paper Morosi and Pizzochero (2006) [16], entirely devoted to the constant $K_{\ell m n d}$ in the special case $\ell = m = n$; many results given therein can be recovered here for this special case.

1. Introduction and preliminaries

The present work generalizes some results of ours [16] on pointwise multiplication in the Sobolev (or Bessel potential) spaces $H^\ell(\mathbf{R}^d, \mathbf{C})$ (see the forthcoming Eqs. (1.38) and (1.39) for a precise definition of these spaces and of their norms). In the cited work, we derived upper and lower bounds for

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the sharp constant $K_{\ell d}$ in the inequality

$$\|fg\|_\ell \leqslant K_{\ell d}\|f\|_\ell\|g\|_\ell \quad \text{for } f, g \in H^\ell\big(\mathbf{R}^d, \mathbf{C}\big),\ \ell > d/2. \tag{1.1}$$

Here, we derive bounds for the sharp constant $K_{\ell mnd}$ in the inequality

$$\begin{gathered}\|fg\|_\ell \leqslant K_{\ell mnd}\|f\|_m\|g\|_n \quad \text{for } f \in H^m\big(\mathbf{R}^d, \mathbf{C}\big),\ g \in H^n\big(\mathbf{R}^d, \mathbf{C}\big),\\ \ell, m, n \in \mathbf{R}, \quad 0 \leqslant \ell \leqslant m \leqslant n, \quad n + m - \ell > d/2;\end{gathered} \tag{1.2}$$

this becomes (1.1) for $\ell = m = n$. The relation $H^m(\mathbf{R}^d, \mathbf{C})H^n(\mathbf{R}^d, \mathbf{C}) \subset H^\ell(\mathbf{R}^d, \mathbf{C})$ and the inequality (1.2) are well known for the indicated values of ℓ, m, n (see e.g. [4, Part 5]); however, to the best of our knowledge, no quantitative analysis seems to have been done for the related constants.

One of the motivations to analyze the constants in this inequality and similar ones is the same indicated in [16]: this analysis allows to infer *a posteriori estimates* on the error of most approximation methods for semilinear evolutionary PDEs with polynomial nonlinearities, and also to get bounds on the time of existence for their exact solutions (see in particular [15], where we considered a nonlinear heat equation and the Navier–Stokes equations). This is just one of the possible applications: in fact, inequalities of the type (1.1), (1.2) and similar ones are relevant for several reasons in many areas of mathematical physics, including the φ^4 quantum field theory and the analysis of the Lieb functional in electronic density theory [10,9].

Let us fix the attention to (1.2). Finding the sharp constant $K_{\ell mnd}$ is clearly difficult; for this reason, and even in view of applications to PDEs, one can be satisfied to derive two-sided bounds

$$K^-_{\ell mnd} \leqslant K_{\ell mnd} \leqslant K^+_{\ell mnd}, \tag{1.3}$$

where the lower bound $K^-_{\ell mnd}$ is sufficiently close to the upper bound $K^+_{\ell mnd}$: this is the same attitude proposed in [16] for the constant $K_{\ell d}$ of (1.1).

In the present paper, we produce the following upper and lower bounds.

(i) First of all, we establish what we call the "$\mathscr{S}$-function" upper bound $K^{\mathscr{S}}_{\ell mnd}$; this is obtained maximizing a suitable function $\mathscr{S}_{\ell mnd} : [0, +\infty) \to (0, +\infty)$ (which is, up to a factor, a generalized hypergeometric function). In the special case $\ell = 0$, we derive as well a "Hölder" upper bound $K^{\mathscr{H}}_{0mnd}$; this is obtained from the Hölder and from the Sobolev imbedding inequalities.

(ii) Next, we present a number of lower bounds; all of them are obtained directly from Eq. (1.2), choosing for f, g some convenient trial functions (generally depending on certain parameters, to be fixed optimally). Different choices of the trial functions yield the so-called "Bessel" lower bound $K^{B_{st}}_{\ell mnd}$, the "Fourier" lower bound $K^F_{\ell mnd}$ and the "S-constant" lower bound $K^S_{\ell\ell nd}$ (holding for $m = \ell$ only).

The above terminology for the upper and lower bounds is used only for convenience: the terms "$\mathscr{S}$-function", etc., recall some distinguished function or feature appearing in the construction of these bounds. For all ℓ, m, n, d, from the available upper and lower bounds one can extract the best ones, indicated with $K^\pm_{\ell mnd}$: so, $K^+_{\ell mnd}$ is the minimum of the upper bounds in (i) and $K^-_{\ell mnd}$ is the maximum of the lower bounds in (ii).

To exemplify the above framework, the paper presents a table of upper and lower bounds $K^\pm_{\ell mnd}$ in dimension $d = 1$ and $d = 3$, for a set of values of ℓ, m, n; in each case, information is provided on the type of bound employed, and on its practical computation. In all cases presented in the table, the ratio $K^-_{\ell mnd}/K^+_{\ell mnd}$ ranges between 0.75 and 1, often reaching a value larger than 0.90; so, our bounds are not far from the sharp constant $K_{\ell mnd}$. It would not be difficult to build similar tables, for different values of ℓ, m, n (even non-integer) and d.

The final step in our analysis is the asymptotics of some available upper and lower bounds, when ℓ, m, n go to infinity (and d is fixed). This generalizes an analysis performed in [16], where we proved for the constant $K_{\ell d}$ in (1.1) the relations

$$0.793T_d\frac{(2/\sqrt{3})^\ell}{\ell^{d/4}}\left[1+O\left(\frac{1}{\ell}\right)\right]\leqslant K_{\ell d}\leqslant T_d\frac{(2/\sqrt{3})^\ell}{\ell^{d/4}}\left[1+O\left(\frac{1}{\ell}\right)\right]\quad\text{for } \ell\to+\infty,$$
$$T_d:=\frac{3^{d/4+1/4}}{2^d\pi^{d/4}} \tag{1.4}$$

(to be intended as follows: $K_{\ell d}$ has upper and lower bounds behaving like the right- and left-hand sides of the above equation).

In the present paper, some of our bounds on the sharp constant $K_{\ell,b\ell,c\ell,d}$ are investigated for $\ell\to+\infty$ and fixed b,c,d ($1\leqslant b\leqslant c$). To exemplify our results, let us report the conclusions arising for $b=c=2$ and $b=2, c=3$, respectively. In the first case we grasp the limiting behavior of the sharp constant, which is the following:

$$K_{\ell,2\ell,2\ell,d}=\frac{1+O(1/\ell)}{(16\pi\ell)^{d/4}}\quad\text{for } \ell\to+\infty; \tag{1.5}$$

the above result is inferred from the analysis of suitable upper and lower bounds for $K_{\ell,2\ell,2\ell,d}$, both of them behaving like the right-hand side of (1.5) when $\ell\to+\infty$.

In the second case, we find

$$\frac{1+O(1/\ell)}{(23\pi\ell)^{d/4}}\leqslant K_{\ell,2\ell,3\ell,d}\underset{(\mathfrak{S}_{23d})}{\leqslant}\frac{1+O(1/\ell)}{(20\pi\ell)^{d/4}}\quad\text{for } \ell\to+\infty. \tag{1.6}$$

The subscript $(\mathfrak{S}_{23d})$ in Eq. (1.6) means that the indicated upper bound holds under a certain condition $\mathfrak{S}_{23d}$, dealing with the maximum of a hypergeometric-like function; we have numerical indications that the condition is satisfied for all d, as explained later in the paper.

Organization of the paper. In the sequel of the present section we fix a few notations, and review some standard properties of the special functions employed throughout the paper (Bessel, hypergeometric, etc.); an integral identity about Bessel functions presented here, and seemingly less trivial, is proved for completeness in Appendix A. Again in this section, we review the definition of the spaces $H^\ell(\mathbf{R}^d,\mathbf{C})$. (Some facts reported in this section were already mentioned in [16]; they have been reproduced to avoid continuous, annoying citation of small details from the previous work.)

In Section 2 we present our upper and lower bounds on $K_{\ell mnd}$, of all the types mentioned before (e.g., the "$\mathscr{S}$-function" upper bound, the "Bessel" lower bound, and so on); most proofs about these bounds are given later, in Sections 5, 6, 7.

In Section 3 we describe the practical computation of the bounds in Section 2, and present the already mentioned table of upper and lower bounds $K^{\pm}_{\ell mnd}$, for $d=1,3$ and many values of ℓ,m,n; further details on the construction of the table are given in Appendix B.

In Section 4 we describe the asymptotics of some upper and lower bounds for $K_{\ell,b\ell,c\ell,d}$, when $1\leqslant b\leqslant c$ and $\ell\to+\infty$; as examples we consider the cases $(b,c)=(2,2)$ and $(2,3)$, yielding the previous mentioned results (1.5) and (1.6). Most statements of Section 4 are proved in Section 8.

Some basic notations and facts. Throughout the paper:

(i) $\mathbf{N}$ stands for $\{0,1,2,\ldots\}$, $\mathbf{N}_0$ means $\mathbf{N}\setminus\{0\}$. We often consider the sets $-\mathbf{N}=\{0,-1,-2,\ldots\}$, $2\mathbf{N}=\{0,2,4,\ldots\}$, $2\mathbf{N}+1=\{1,3,5,\ldots\}$ and $\mathbf{N}+\frac{1}{2}=\{\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots\}$.

(ii) We use the double factorial

$$(-1)!!:=1;\qquad s!!:=1\cdot 3\cdot\ldots\cdot(s-2)s\quad\text{for } s\in 2\mathbf{N}+1. \tag{1.7}$$

(iii) The Pochhammer symbol of $a \in \mathbf{R}$, $i \in \mathbf{N}$ is

$$(a)_i := 1 \quad \text{if } i = 0, \qquad (a)_i := a(a+1)\dots(a+i-1) \quad \text{if } i > 0; \tag{1.8}$$

note that

$$(-s)_i = 0 \quad \text{for } s \in \mathbf{N},\ i > s. \tag{1.9}$$

(iv) We work in any space dimension $d \in \mathbf{N}_0$. The standard inner product and Euclidean norm of $\mathbf{R}^d$ are denoted by $\bullet$ and $|\ |$, respectively. The running variable over $\mathbf{R}^d$ is written $x = (x^1, \dots, x^d)$ (or k, when $\mathbf{R}^d$ is viewed as the space of "wave vectors" for the Fourier transform); the Lebesgue measure of $\mathbf{R}^d$ is indicated with dx (or dk).

For future citation, we record here the familiar formula for integrals over $\mathbf{R}^d$ of radially symmetric functions; this is the equation

$$\int_{\mathbf{R}^d} dx\, \varphi\big(|x|\big) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} dr\, r^{d-1} \varphi(r), \tag{1.10}$$

holding for all sufficiently regular real (or complex) functions φ on $(0, +\infty)$ (when dealing with integrals on the "wave vector" space $(\mathbf{R}^d, dk)$, the radius r is renamed ρ).

Some special functions. The independent variables and the parameters appearing in the special functions that we consider are *real*, unless the use of complex numbers is explicitly declared; consequently, the notion of analyticity often employed in relation with such functions is intended in the real sense. We take [6] as a general reference on real analyticity; in particular, we frequently refer to the principle of analytic continuation as stated in Corollary 2, page 122 of the cited book.

We take [1,11,17,19] as standard references for special functions. In this paper, we frequently use: the Gamma function Γ; the Bessel functions of the first kind J_ν, the modified Bessel functions of the first kind I_ν and the modified Bessel functions of the second kind, or Macdonald functions, K_ν; the generalized hypergeometric functions ${}_pF_q$, especially in the cases $p = 2, q = 1$ (the usual Gaussian hypergeometric function) and $p = 3, q = 2$.

Concerning the Gamma function, we often use: the integral representation

$$\Gamma(\alpha) = \int_0^{+\infty} dp\, p^{\alpha-1} e^{-p} \quad \text{for } \alpha \in (0, +\infty), \tag{1.11}$$

the elementary relations

$$\Gamma(k+1) = k!, \qquad \Gamma(\alpha + k) = (\alpha)_k \Gamma(\alpha) \quad \text{for } k \in \mathbf{N}, \tag{1.12}$$

the duplication formula

$$\Gamma(2\alpha) = \frac{2^{2\alpha-1}}{\sqrt{\pi}} \Gamma\left(\alpha + \frac{1}{2}\right) \Gamma(\alpha), \tag{1.13}$$

the integral identity

$$\int_0^1 dt\, t^{\alpha-1}(1-t)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{for } \alpha, \beta \in (0, +\infty), \tag{1.14}$$

and the asymptotics

$$\frac{\Gamma(\alpha+\mu)}{\Gamma(\alpha+\nu)} = \alpha^{\mu-\nu}\left[1 + O\left(\frac{1}{\alpha}\right)\right] \quad \text{for } \mu, \nu \in \mathbf{R},\ \alpha \to +\infty. \tag{1.15}$$

As for the Macdonald functions, we recall that

$$K_\nu(w) = \sqrt{\pi}\, e^{-w} \sum_{i=0}^{\nu-1/2} \frac{(2\nu-i-1)!}{i!(\nu-i-1/2)!} \frac{1}{(2w)^{\nu-i}} \quad \text{for } \nu \in \mathbf{N} + \frac{1}{2},\ w \in \mathbf{R}. \tag{1.16}$$

The list of results we need about ${}_pF_q$ functions is longer, and wholly occupies the next paragraph.

On (generalized) hypergeometric functions. Most of the facts reported hereafter on the ${}_pF_q$ hypergeometric functions are derived from [11]; we will occasionally mention other references. Let

$$p, q \in \mathbf{N}, \quad \alpha_1, \dots, \alpha_p \in \mathbf{R}, \quad \delta_1, \dots, \delta_q \in \mathbf{R} \setminus (-\mathbf{N}); \tag{1.17}$$

for $k = 0, 1, 2, \dots$ we associate to the parameters $\alpha_1, \dots, \delta_q$ the Pochhammer's symbols $(\alpha_1)_k, \dots,$ $(\alpha_p)_k, (\delta_1)_k, \dots, (\delta_q)_k$, noting that $(\delta_i)_k \neq 0$ due to the assumptions on δ_i. If w is a real variable, the standard definition

$${}_pF_q(\alpha_1, \dots, \alpha_p; \delta_1, \dots, \delta_q; w) := \sum_{k=0}^{+\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\delta_1)_k \dots (\delta_q)_k} \frac{w^k}{k!} \tag{1.18}$$

makes sense when the above power series in w converges; this happens, in particular, if

$$p = q, \quad w \in \mathbf{R} \tag{1.19}$$

or

$$p = q+1, \quad w \in (-1, 1), \tag{1.20}$$

or

$$p, q \text{ arbitrary},\ \alpha_i = -\ell \text{ for some } i \in \{1, \dots, p\} \text{ and } \ell \in \mathbf{N},\ w \in \mathbf{R}; \tag{1.21}$$

in the third case we have $(\alpha_i)_k = 0$ for $k > \ell$, so the series $\sum_{k=0}^{+\infty}$ in (1.18) is in fact a finite sum $\sum_{k=0}^{\ell}$. In the subcase $\ell = 0$ of (1.21), the finite sum consists only of the $k = 0$ term, so

$${}_pF_q(\alpha_1, \dots, \alpha_p; \delta_1, \dots, \delta_q; w) = 1$$
$$\text{for } p, q \text{ arbitrary, if } \alpha_i = 0 \text{ for some } i \in \{1, \dots, p\} \text{ and } w \in \mathbf{R}. \tag{1.22}$$

In general, the series (1.18) is invariant under arbitrary permutations of the parameters $\alpha_1, \dots, \alpha_p$ or $\delta_1, \dots, \delta_q$.

Due to the above indications on the case $p = q$, the function ${}_qF_q(\alpha_1, \ldots, \alpha_q; \delta_1, \ldots, \delta_q; w)$ is well defined via (1.18) for

$$\alpha_1, \ldots, \alpha_q \in \mathbf{R}, \quad \delta_1, \ldots, \delta_q \in \mathbf{R} \setminus (-\mathbf{N}), \quad w \in \mathbf{R}; \tag{1.23}$$

furthermore, ${}_qF_q$ is analytic in all the parameters α_i, δ_i and in the variable w on the domain (1.23). For fixed $\alpha_1, \ldots, \delta_q$ as in (1.23), one has ${}_qF_q(\alpha_1, \ldots, \alpha_q, \delta_1, \ldots, \delta_q; w) = O((-w)^{-\mu})$ for $w \to -\infty$, and ${}_qF_q(\alpha_1, \ldots, \alpha_q, \delta_1, \ldots, \delta_q; w) = O(w^{\nu} e^{w})$ for $w \to +\infty$, where $\mu := \min(\alpha_1, \ldots, \alpha_q)$, $\nu := \sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{q} \delta_i$; these results can be traced in the classical work [3].

Concerning the case $p = q + 1$, the limitation $w \in (-1, 1)$ in Eq. (1.20) can be overcome if at least one of the parameters $\alpha_1, \ldots, \alpha_{q+1}$ is positive; in this case, one can define ${}_{q+1}F_q$ using, instead of the series (1.18), the following integral formula (see [11, vol. I, p. 59, Eq. (13)]):

$$\begin{aligned}
&{}_{q+1}F_q(\alpha_1, \ldots, \alpha_{q+1}; \delta_1, \ldots, \delta_q; w) \\
&:= \frac{1}{\Gamma(\alpha_h)} \int_0^{+\infty} dt\, e^{-t} t^{\alpha_h - 1} {}_qF_q(\alpha_1, \ldots, \alpha_{h-1}, \alpha_{h+1}, \ldots, \alpha_{q+1}; \delta_1, \ldots, \delta_q; wt) \\
&\text{if } \alpha_h \in (0, +\infty) \text{ for some } h \in \{1, \ldots, q+1\} \text{ and } \alpha_1, \ldots, \alpha_{h-1}, \alpha_{h+1}, \ldots, \alpha_{q+1} \in \mathbf{R}, \\
&\delta_1, \ldots, \delta_q \in \mathbf{R} \setminus (-\mathbf{N}), \quad w \in (-\infty, 1).
\end{aligned} \tag{1.24}$$

The above integral converges, due to the previous result on the asymptotics of ${}_qF_q$ for large values of the variable. The prescription (1.24) gives a unique definition for ${}_{q+1}F_q$ if applied for different values of h (all of them with $\alpha_h > 0$), and always agrees with Eq. (1.18) if $w \in (-1, 1)$, or if $\alpha_i = -s$ for some $i \in \{1, \ldots, p\}$, $s \in \mathbf{N}$ and $w \in (-\infty, 1)$.

The function ${}_{q+1}F_q$ is analytic in the parameters $\alpha_1, \ldots, \alpha_{q+1}, \delta_1, \ldots, \delta_q$ and in the variable w in the domain indicated by Eqs. (1.20), (1.21) and (1.24). Of course, many properties of ${}_{q+1}F_q$ derivable where the series (1.18) converges hold in fact on the whole domain (1.20), (1.21) and (1.24), by the principle of analytic continuation.

Let us finally mention that, for $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$,

$$\begin{aligned}
&{}_{p+1}F_{q+1}(\alpha_1, \ldots, \alpha_{i-1}, \beta, \alpha_i, \ldots, \alpha_p; \delta_1, \ldots, \delta_{j-1}, \beta, \delta_j, \ldots, \delta_q; w) \\
&\quad = {}_pF_q(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i, \ldots, \alpha_p; \delta_1, \ldots, \delta_{j-1}, \delta_j, \ldots, \delta_q; w)
\end{aligned} \tag{1.25}$$

whenever the two sides are defined (by power series of the type (1.18), or by any analytic continuation).

As anticipated, in this paper we are mainly interested in the ${}_2F_1$ and ${}_3F_2$ hypergeometric functions.

The properties of ${}_2F_1(\alpha, \beta; \delta; w)$ we are using more frequently are the obvious symmetry in α, β, and the Kummer transformation

$${}_2F_1(\alpha, \beta; \delta; w) = (1 - w)^{\delta - \alpha - \beta} {}_2F_1(\delta - \alpha, \delta - \beta; \delta; w). \tag{1.26}$$

Besides the integral representation (1.24), we have for this function the alternative representations

$$\begin{aligned}
{}_2F_1(\alpha, \beta; \delta; w) &= \frac{\Gamma(\delta)}{\Gamma(\beta)\Gamma(\delta - \beta)} \int_0^1 ds\, s^{\beta - 1} (1 - s)^{\delta - \beta - 1} (1 - sw)^{-\alpha} \\
&\qquad \text{for } \delta > \beta > 0,\ -\infty < w < 1;
\end{aligned} \tag{1.27}$$

$$
{}_2F_1(\alpha,\beta;\delta;1-w)=\frac{\Gamma(\delta)}{\Gamma(\beta)\Gamma(\delta-\beta)}\int_0^{+\infty} du\, u^{\beta-1}(1+u)^{\alpha-\delta}(1+wu)^{-\alpha}>0
$$
$$
\text{for } \delta>\beta>0,\ w>0. \tag{1.28}
$$

Eq. (1.27) is the well-known Euler's formula, and (1.28) follows from (1.27) after a change of variable $s=u/(1+u)$.

The function ${}_3F_2(\alpha,\beta,\gamma;\delta,\epsilon;\eta)$ is obviously symmetric in α,β,γ and δ,ϵ separately. In the sequel we refer to the identity (see [11, vol. II, p. 13, Eq. (34)])

$$
{}_3F_2(\alpha,\beta,\gamma;\delta,\epsilon;w)=\sum_{i=0}^{+\infty}\frac{(\alpha)_i(\beta)_i(\epsilon-\gamma)_i}{(\delta)_i(\epsilon)_i}\frac{(-w)^i}{i!}{}_2F_1(\alpha+i,\beta+i;\delta+i;w)
$$
$$
\text{for } -\infty<w<\frac{1}{2}. \tag{1.29}
$$

We also mention the asymptotics [8,18]

$$
{}_2F_1(\alpha,\beta;\delta;w)\sim\frac{\Gamma(\beta-\alpha)\Gamma(\delta)}{\Gamma(\delta-\alpha)\Gamma(\beta)}(-w)^{-\alpha}
$$
$$
\text{for } w\to-\infty,\ \beta,\delta>0,\ \alpha<\min(\beta,\delta); \tag{1.30}
$$
$$
{}_3F_2(\alpha,\beta,\gamma;\delta,\epsilon;w)\sim\frac{\Gamma(\delta)\Gamma(\epsilon)\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta-\alpha)\Gamma(\epsilon-\alpha)}(-w)^{-\alpha}
$$
$$
\text{for } w\to-\infty,\ \beta,\gamma,\delta,\epsilon>0,\ \alpha<\min(\beta,\gamma,\delta,\epsilon). \tag{1.31}
$$

Another result, important for our purposes, is the relation

$$
\begin{aligned}
&\int_0^{+\infty} dr\, r^{\mu+\nu+\delta+1}J_\delta(hr)K_\mu(r)K_\nu(r)\\
&=2^{\mu+\nu+\delta-1}\frac{\Gamma(\mu+\delta+1)\Gamma(\nu+\delta+1)\Gamma(\mu+\nu+\delta+1)}{\Gamma(\mu+\nu+2\delta+2)}h^\delta\\
&\quad\times{}_3F_2\left(\mu+\delta+1,\nu+\delta+1,\mu+\nu+\delta+1;\frac{\mu+\nu}{2}+\delta+1,\frac{\mu+\nu}{2}+\delta+\frac{3}{2};-\frac{h^2}{4}\right)\\
&\text{for } h,\mu,\nu,\delta\in\mathbf{R},\ h>0,\ \delta,\mu+\delta,\nu+\delta,\mu+\nu+\delta>-1;
\end{aligned} \tag{1.32}
$$

the above conditions on the parameters ensure, amongst else, convergence of the integral in the left-hand side. Eq. (1.32) generalizes Eq. (3.16) of [16], and the considerations of the cited reference can be rephrased in the present framework: the result (1.32) is known, but it is difficult to trace a proof in the literature. For this reason, a derivation of (1.32) is proposed in Appendix A.

Fourier transform. Let us use the standard notation $S'(\mathbf{R}^d,\mathbf{C})$ for the tempered distributions on $\mathbf{R}^d$. We denote with $\mathcal{F},\mathcal{F}^{-1}:S'(\mathbf{R}^d,\mathbf{C})\to S'(\mathbf{R}^d,\mathbf{C})$ the Fourier transform and its inverse; $\mathcal{F}$ is normalized so that

$$
\mathcal{F}f(k)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbf{R}^d} dx\, e^{-ik\bullet x}f(x) \tag{1.33}
$$

(intending the integral literally, if $f \in L^1(\mathbf{R}^d, \mathbf{C})$). The restriction of $\mathcal{F}$ to $L^2(\mathbf{R}^d, \mathbf{C})$, with the standard inner product and the associated norm $\| \, \|_{L^2}$, is a Hilbertian isomorphism.

Consider two (sufficiently regular) radially symmetric functions

$$f : \mathbf{R}^d \to \mathbf{C}, \quad x \to f(x) = \varphi\big(|x|\big), \quad F : \mathbf{R}^d \to \mathbf{C}, \quad k \to F(k) = \Phi\big(|k|\big); \tag{1.34}$$

the Fourier and inverse Fourier transforms $\mathcal{F} f$, $\mathcal{F}^{-1} F$ are also radially symmetric, and given by [5]

$$\mathcal{F} f(k) = \frac{1}{|k|^{d/2-1}} \int_0^{+\infty} dr\, r^{d/2} J_{d/2-1}\big(|k| r\big) \varphi(r), \tag{1.35}$$

$$\mathcal{F}^{-1} F(x) = \frac{1}{|x|^{d/2-1}} \int_0^{+\infty} d\rho\, \rho^{d/2} J_{d/2-1}\big(|x| \rho\big) \Phi(\rho). \tag{1.36}$$

Sobolev spaces. Let us consider a real number ℓ; we denote with $\sqrt{1+|\mathrm{k}|^2}^{\,\ell}$ the function $k \in \mathbf{R}^d \mapsto \sqrt{1+|k|^2}^{\,\ell}$ (and the multiplication operator by this function). Furthermore, we put

$$\sqrt{1-\Delta}^{\,\ell} := \mathcal{F}^{-1} \sqrt{1+|\mathrm{k}|^2}^{\,\ell} \mathcal{F} : S'\big(\mathbf{R}^d, \mathbf{C}\big) \to S'\big(\mathbf{R}^d, \mathbf{C}\big). \tag{1.37}$$

The ℓ-th order Sobolev (or Bessel potential) space of L^2-type and its norm are [2,12]

$$\begin{aligned} H^\ell\big(\mathbf{R}^d, \mathbf{C}\big) &:= \big\{ f \in S'\big(\mathbf{R}^d, \mathbf{C}\big) \bigm| \sqrt{1-\Delta}^{\,\ell} f \in L^2\big(\mathbf{R}^d, \mathbf{C}\big) \big\} \\ &= \big\{ f \in S'\big(\mathbf{R}^d, \mathbf{C}\big) \bigm| \sqrt{1+|\mathrm{k}|^2}^{\,\ell} \mathcal{F} f \in L^2\big(\mathbf{R}^d, \mathbf{C}\big) \big\}; \end{aligned} \tag{1.38}$$

$$\| f \|_\ell := \big\| \sqrt{1-\Delta}^{\,\ell} f \big\|_{L^2} = \big\| \sqrt{1+|\mathrm{k}|^2}^{\,\ell} \mathcal{F} f \big\|_{L^2}. \tag{1.39}$$

We note the equality

$$\big(H^0\big(\mathbf{R}^d\big), \| \, \|_0\big) = \big(L^2\big(\mathbf{R}^d\big), \| \, \|_{L^2}\big) \tag{1.40}$$

and the imbedding relations

$$\ell \leqslant \ell' \quad \Rightarrow \quad H^{\ell'}(\mathbf{R})^d \subset H^\ell\big(\mathbf{R}^d\big), \quad \| \, \|_\ell \leqslant \| \, \|_{\ell'}. \tag{1.41}$$

We only consider the Sobolev spaces $H^\ell(\mathbf{R}^d)$ of order $\ell \geqslant 0$, which are embedded into $L^2(\mathbf{R}^d)$ (and so, consist of ordinary functions). In the special case $\ell \in \mathbf{N}$, the definitions (1.38) and (1.39) imply

$$\begin{aligned} H^\ell\big(\mathbf{R}^d, \mathbf{C}\big) = \big\{ f \in S'\big(\mathbf{R}^d, \mathbf{C}\big) \bigm| & \partial_{\lambda_1, \ldots, \lambda_k} f \in L^2\big(\mathbf{R}^d, \mathbf{C}\big) \\ & \forall k \in \{0, \ldots, \ell\},\ (\lambda_1, \ldots, \lambda_k) \in \{1, \ldots, d\}^k \big\}; \end{aligned} \tag{1.42}$$

$$\| f \|_\ell = \sqrt{\sum_{k=0}^{\ell} \binom{\ell}{k} \sum_{\lambda_1, \ldots, \lambda_k = 1, \ldots, d} \int_{\mathbf{R}^d} dx\, \big| \partial_{\lambda_1, \ldots, \lambda_k} f(x) \big|^2}. \tag{1.43}$$

In the above, ∂_{λ_i} is the distributional derivative with respect to the coordinate x^{λ_i}.

Other functions. As in [16], a central role in our considerations is played by the function $G_{td} := 1/(1+|k|^2)^t$, i.e.,

$$G_{td} : \mathbf{R}^d \to \mathbf{C}, \quad k \mapsto G_{td}(k) := \frac{1}{(1+|k|^2)^t} \quad (t \in \mathbf{R}); \tag{1.44}$$

we further set

$$g_{td} : \mathbf{R}^d \to \mathbf{C}, \quad g_{td} := \mathcal{F}^{-1} G_{td} \quad (t > d/4). \tag{1.45}$$

We note that, with the assumption $t > d/4$, G_{td} and, consequently, g_{td} are L^2 functions. The functions g_{td} are related to the Macdonald functions [2,12] since, for any $x \in \mathbf{R}^d$,

$$g_{td}(x) = \frac{|x|^{t-d/2}}{2^{t-1}\Gamma(t)} K_{t-d/2}\big(|x|\big). \tag{1.46}$$

2. The constant $K_{\ell mnd}$ and its bounds: description of the main results

Let $d \in \mathbf{N}_0$, and consider three real numbers ℓ, m, n such that

$$0 \leqslant \ell \leqslant m \leqslant n, \quad n + m - \ell > d/2. \tag{2.1}$$

Definition 2.1. We put

$$K_{\ell mnd} := \min\big\{K \in [0, +\infty) \;\big|\; \|fg\|_\ell \leqslant K \|f\|_m \|g\|_n \text{ for all } f \in H^m\big(\mathbf{R}^d, \mathbf{C}\big),\ g \in H^n\big(\mathbf{R}^d, \mathbf{C}\big)\big\} \tag{2.2}$$

and refer to this as the sharp (or best) constant for the multiplication $H^m(\mathbf{R}^d, \mathbf{C}) \times H^n(\mathbf{R}^d, \mathbf{C}) \to H^\ell(\mathbf{R}^d, \mathbf{C})$.

In the sequel we present our upper and lower bounds for the above constant; most of the forthcoming propositions are proved in Sections 5, 6, 7.

"$\mathscr{S}$-function" upper bound on $K_{\ell mnd}$. This is our most important upper bound; it is determined by a function $\mathscr{S} = \mathscr{S}_{\ell mnd}$, as stated hereafter.

Proposition 2.2. (i) *For ℓ, m, n fulfilling* (2.1), *one has*

$$K_{\ell mnd} \leqslant \sqrt{\sup_{u \in [0,+\infty)} \mathscr{S}_{\ell mnd}(u)}, \tag{2.3}$$

where, for $u \in [0, +\infty)$,

$$\mathscr{S}_{\ell mnd}(u) := \frac{\Gamma(m+n-d/2)}{(4\pi)^{d/2}\Gamma(n+m)} (1+4u)^\ell F_{mnd}(u), \tag{2.4}$$

$$F_{mnd}(u) := {}_3F_2\left(m+n-\frac{d}{2}, m, n; \frac{m+n}{2}, \frac{m+n+1}{2}; -u\right). \tag{2.5}$$

In the special case $m = n$, Eq. (2.5) *implies*

$$F_{mmd}(u) = {}_2F_1\left(2m - \frac{d}{2}, m; m + \frac{1}{2}; -u\right); \tag{2.6}$$

the trivial case $m = 0$ is described by

$$F_{0nd}(u) = 1 \quad \textit{for all } u. \tag{2.7}$$

For all ℓ, m, n as in (2.1)*, the function $\mathscr{S}_{\ell mnd}$ sends* $[0, +\infty)$ *to* $(0, +\infty)$ *and is bounded, so the sup in* (2.3) *is actually finite. The behavior of this function for $u = 0$ and $u \to +\infty$ is described by the following relations*:

$$\mathscr{S}_{\ell mnd}(0) = \frac{\Gamma(m+n-d/2)}{(4\pi)^{d/2}\Gamma(n+m)}, \tag{2.8}$$

$$\mathscr{S}_{\ell mnd}(u) \sim \frac{(1+\delta_{mn})\Gamma(n-d/2)}{(4\pi)^{d/2}\Gamma(n)} \frac{1}{(4u)^{m-\ell}} \quad \textit{for } u \to +\infty \tag{2.9}$$

(δ *is the Kronecker symbol, i.e.*, $\delta_{mn} := 1$ *if* $m = n$*, and* $\delta_{mn} := 0$ *otherwise*)*. According to* (2.9)*, the $u \to +\infty$ limit of $\mathscr{S}_{\ell mnd}$ is*

$$\mathscr{S}_{\ell mnd}(+\infty) = \begin{cases} \frac{(1+\delta_{mn})\Gamma(n-d/2)}{(4\pi)^{d/2}\Gamma(n)} & \textit{if } \ell = m, \\ 0 & \textit{if } \ell < m. \end{cases} \tag{2.10}$$

(ii) *One has*

$$F_{mnd}(u) = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} \frac{(m+n-\frac{d}{2})_i (m)_i (\frac{m-n+1}{2})_i}{i!(\frac{m+n+1}{2})_i(\frac{m+n}{2})_i} \frac{(\frac{d-m-n}{2})_j(\frac{n-m}{2})_j}{j!(\frac{m+n}{2}+i)_j} \frac{(-1)^j u^{i+j}}{(1+u)^{\frac{3m+n-d}{2}+i}} \tag{2.11}$$

if $u \in [0, 1)$, or $u \in [0, +\infty)$ and the series over j is a finite sum.

An alternative expansion, holding under the same conditions, is

$$F_{mnd}(u) = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} \frac{(m+n-\frac{d}{2})_i (m)_i (\frac{m-n}{2})_i}{i!(\frac{m+n+1}{2})_i(\frac{m+n}{2})_i} \frac{(\frac{d+1-m-n}{2})_j(\frac{n+1-m}{2})_j}{j!(\frac{m+n+1}{2}+i)_j} \frac{(-1)^j u^{i+j}}{(1+u)^{\frac{3m+n-d-1}{2}+i}}. \tag{2.12}$$

The above series over j or i become finite sums in the special cases indicated below:

$$\textit{If } m+n-d \in 2\mathbf{N}, \quad \sum_{j=0}^{+\infty} \to \sum_{j=0}^{\frac{m+n-d}{2}} \quad \textit{in } (2.11);$$
$$\textit{if } n-m \in 2\mathbf{N}+1, \quad \sum_{i=0}^{+\infty} \to \sum_{i=0}^{\frac{n-m-1}{2}} \quad \textit{in } (2.11). \tag{2.13}$$

$$\textit{If } m+n-d \in 2\mathbf{N}+1, \quad \sum_{j=0}^{+\infty} \to \sum_{j=0}^{\frac{m+n-d-1}{2}} \quad \textit{in } (2.12);$$
$$\textit{if } n-m \in 2\mathbf{N}, \quad \sum_{i=0}^{+\infty} \to \sum_{i=0}^{\frac{n-m}{2}} \quad \textit{in } (2.12). \tag{2.14}$$

Proof. See Section 5. □

Remark 2.3. In the case $\ell = m = n$ ($\ell > d/2$), Eqs. (2.4)–(2.6) give

$$\mathscr{S}_{\ell\ell\ell d}(u) := \frac{\Gamma(2\ell - d/2)}{(4\pi)^{d/2}\Gamma(2\ell)}(1+4u)^{\ell}\,{}_2F_1\left(2\ell - \frac{d}{2}, \ell; \ell + \frac{1}{2}; -u\right); \tag{2.15}$$

this is the function denoted with $\mathscr{S}_{\ell d}$ in [16, Proposition 2.2], that was employed to derive our upper bound on $K_{\ell\ell\ell d} \equiv K_{\ell d}$.

"Hölder" upper bound on K_{0mnd}. The upper bound on $K_{\ell mnd}$ given by the above proposition holds for arbitrary ℓ, m, n as in (2.1). In this paragraph we give a different upper bound for the special case $\ell = 0$, that is somehow trivial since $\| \; \|_0$ is the L^2-norm. In this case, for all functions f, g one can estimate $\|fg\|_{L^2}$ via the Hölder inequality and then employ the Sobolev imbedding inequality, with certain information on the related constant.

Let us recall the classical Sobolev imbedding theorem: this states that, for $t \in [0, d/2)$, $r \in [2, \frac{d}{d/2-t}]$, or $t = d/2$, $r \in [2, +\infty)$, or $t \in (d/2, +\infty)$, $r \in [2, +\infty]$, one has $H^t(\mathbf{R}^d) \subset L^r(\mathbf{R}^d)$, and $\| \; \|_{L^r} \leqslant S \| \; \|_t$ for some $S \in [0, +\infty)$.

Some information on the constants S was given in our previous work [13]; here we considered the set

$$R_{td} := \begin{cases} \{2\} & \text{if } t = 0, \\ [2, \frac{d}{d/2-t}) & \text{if } t \in (0, d/2), \\ [2, +\infty) & \text{if } t = d/2, \\ [2, +\infty] & \text{if } t \in (d/2, +\infty) \end{cases} \tag{2.16}$$

and proved the following:

$$t \in [0, +\infty), \quad r \in R_{td} \quad \Rightarrow \quad \| \; \|_{L^r} \leqslant S_{rtd} \| \; \|_t, \tag{2.17}$$

where

$$S_{rtd} := \frac{1}{(4\pi)^{d/4 - d/(2r)}}\left(\frac{\Gamma(\frac{t}{1-2/r} - \frac{d}{2})}{\Gamma(\frac{t}{1-2/r})}\right)^{1/2-1/r}\left(\frac{E(1/r)}{E(1-1/r)}\right)^{d/2}$$

$$\text{if } t \in [0, d/2),\ r \in \left(2, \frac{d}{d/2-t}\right) \quad \text{or} \quad t \in [d/2, +\infty), r \in (2, +\infty), \tag{2.18}$$

$$S_{2td} := 1 \quad \text{if } t \in [0, +\infty), \tag{2.19}$$

$$S_{\infty td} := \frac{1}{(4\pi)^{d/4}}\sqrt{\frac{\Gamma(t - d/2)}{\Gamma(t)}} \quad \text{if } t \in (d/2, +\infty); \tag{2.20}$$

$$E(u) := u^u \quad \text{for } u \in (0, +\infty), \qquad E(0) := \lim_{u \to 0^+} E(u) = 1. \tag{2.21}$$

As a matter of fact, S_{rtd} is the best constant fulfilling the imbedding inequality in the case $r = +\infty$:

$$S_{\infty td} := \min\left\{S \in [0, +\infty) \;\middle|\; \| \; \|_{L^\infty(\mathbf{R}^d)} \leqslant S \| \; \|_t\right\} \quad \text{for } t \in \left(\frac{d}{2}, +\infty\right); \tag{2.22}$$

to prove this, in [13] we have shown that the equality $\|f\|_{L^\infty(\mathbf{R}^d)} = S_{\infty td}\|f\|_t$ holds for $f = g_{td}$ as in Eqs. (1.45) and (1.46).

With the previous notations, we can state the following.

Proposition 2.4. *For any $p \in [2, +\infty]$, let $p^* \in [2, +\infty]$ denote the solution of the equation $1/p + 1/p^* = 1/2$. Furthermore, let m, n fulfill conditions* (2.1)*, with $\ell = 0$; then,* (i) *and* (ii) *hold.*

(i) *The set*

$$R_{mnd} := \{ p \in R_{md} \mid p^* \in R_{nd} \} \tag{2.23}$$

is nonempty.

(ii) *For any $p \in R_{mnd}$, one has*

$$K_{0mnd} \leqslant S_{pmd} S_{p^*nd}; \tag{2.24}$$

so,

$$K_{0mnd} \leqslant \inf_{p \in R_{mnd}} S_{pmd} S_{p^*nd}. \tag{2.25}$$

Proof. (i) The thesis follows from an elementary analysis, explicitating the definitions of R_{md} and R_{nd} via Eq. (2.16).

(ii) Let $p \in R_{mnd}$, and consider any two functions $f \in H^m(\mathbf{R}^d)$, $g \in H^n(\mathbf{R}^d)$; then, the Hölder inequality and the imbedding relations (2.17) give

$$\|fg\|_0 = \|fg\|_{L^2} \leqslant \|f\|_{L^p} \|g\|_{L^{p^*}} \leqslant \big(S_{pmd}\|f\|_m\big)\big(S_{p^*nd}\|g\|_n\big), \tag{2.26}$$

whence the thesis (2.24). Now, (2.25) is obvious. □

As shown later via a series of examples, the bound (2.25) is often better than the case $\ell = 0$ of the bound (2.3).

General method to get lower bounds on $K_{\ell mnd}$. The general method is based on the obvious inequality

$$K_{\ell mnd} \geqslant \frac{\|fg\|_\ell}{\|f\|_m \|g\|_n} \tag{2.27}$$

for all nonzero $f \in H^m(\mathbf{R}^d, \mathbf{C})$, $g \in H^n(\mathbf{R}^d, \mathbf{C})$; this gives a lower bound for any pair of "trial functions" f, g. In the sequel we propose several choices of the trial functions, depending on one or more parameters; the parameters must be tuned to get the best lower bound, i.e., the maximum value for the right-hand side of Eq. (2.27).

"Bessel" lower bound. In this approach, the trial functions have the form

$$g_{\nu td}(x) := g_{td}(\nu x) \tag{2.28}$$

where $\nu \in (0, +\infty)$ is a parameter and g_{td} is defined by Eq. (1.45). By comparison with that equation, we find

$$g_{\nu td} = \mathcal{F}^{-1} G_{\nu td}, \qquad G_{\nu td}(k) := \frac{1}{\nu^d (1 + |k|^2/\nu^2)^t}. \tag{2.29}$$

Proposition 2.5. (i) *Let $n \in [0, +\infty)$, $t \in (n/2 + d/4, +\infty)$, $\nu \in (0, +\infty)$. Then*

$$g_{\nu td} \in H^n\big(\mathbf{R}^d, \mathbf{C}\big),$$

$$\|g_{\nu td}\|_n^2 = \frac{\pi^{d/2}}{\nu^d} \frac{\Gamma(2t - n - d/2)}{\Gamma(2t - n)} {}_2F_1\big(-n, d/2; 2t - n; 1 - \nu^2\big). \tag{2.30}$$

(*Note that ${}_2F_1(-n, d/2; 2t - n; w)$ is a finite sum $\sum_{i=0}^{n} \frac{(-n)_i (d/2)_i}{(2t-n)_i} \frac{w^i}{i!}$ if $n \in \mathbf{N}$.*)

(ii) *Let ℓ, m, n fulfill* (2.1)*, and*

$$s \in (m/2 + d/4, +\infty), \quad t \in (n/2 + d/4, +\infty), \quad \mu, \nu \in (0, +\infty) \tag{2.31}$$

(*then $g_{\mu sd} \in H^m(\mathbf{R}^d, \mathbf{C})$ and $g_{\nu td} \in H^n(\mathbf{R}^d, \mathbf{C})$, due to* (i)*; this also implies $g_{\mu sd} g_{\nu td} \in H^\ell(\mathbf{R}^d, \mathbf{C})$). One has*

$$\|g_{\mu sd} g_{\nu td}\|_\ell^2 = \frac{2^d \pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} du\, u^{d/2-1} (1 + 4u)^\ell G_{std}^2(\mu, \nu; u), \tag{2.32}$$

where

$$G_{std}(\mu, \nu; u) := \frac{\mu^{s-d/2} \nu^{t-d/2}}{2^{2s+2t-2} \Gamma(s) \Gamma(t) u^{s/2+t/2}} \times \int_0^{+\infty} dr\, r^{s+t-d/2} J_{d/2-1}(r) K_{s-d/2}\left(\frac{\mu r}{2\sqrt{u}}\right) K_{t-d/2}\left(\frac{\nu r}{2\sqrt{u}}\right). \tag{2.33}$$

Moreover, assume

$$s - \frac{d}{2}, t - \frac{d}{2} \in \mathbf{N} + \frac{1}{2}, \quad \ell \in \mathbf{N}. \tag{2.34}$$

Then both integrals in Eqs. (2.33) *and* (2.32) *are elementary, and*

$$\begin{aligned} \|g_{\mu sd} g_{\nu td}\|_\ell^2 = {} & \frac{\pi^{d/2+2}}{\Gamma^3(d/2) \Gamma^2(s) \Gamma^2(t)} \sum_{h=0}^{\ell} \sum_{(i,j,k) \in I_{std}} \sum_{(i',j',k') \in I_{std}} \binom{\ell}{h} \\ & \times \frac{\Gamma(i + i' + j + j' - k - k' - h + d/2 + 1) \Gamma(k + k' + h + d/2)}{\Gamma(i + i' + j + j' + d + 1)} G_{stijkd} G_{sti'j'k'd} \\ & \times \frac{\mu^{i+i'} \nu^{j+j'}}{(\mu + \nu)^{i+i'+j+j'-2h+d}}. \end{aligned} \tag{2.35}$$

Here we have put

$$I_{std} := \left\{ (i, j, k) \in \mathbf{N}^3 \,\middle|\, 0 \leqslant i \leqslant s - \frac{d}{2} - \frac{1}{2},\ 0 \leqslant j \leqslant t - \frac{d}{2} - \frac{1}{2},\ 0 \leqslant k \leqslant \frac{i + j + 1}{2} \right\}, \tag{2.36}$$

$$G_{stijkd} := \frac{(-1)^k (i + j + d - 1)! (2s - i - d - 1)! (2t - j - d - 1)! (-\frac{i+j}{2})_k (-\frac{i+j+1}{2})_k}{2^{2s+2t-i-j-d/2-3} i! j! k! (s - i - \frac{d}{2} - \frac{1}{2})! (t - j - \frac{d}{2} - \frac{1}{2})! (\frac{d}{2})_k}. \tag{2.37}$$

(iii) *Let ℓ, m, n be as in* (2.1)*, and s, t as in* (ii)*. Then, for all $\mu, \nu \in (0, +\infty)$,*

$$K_{\ell mnd} \geqslant \mathscr{K}^{B_{st}}_{\ell mnd}(\mu, \nu) := \frac{\|g_{\mu sd} g_{\nu td}\|_\ell}{\|g_{\mu sd}\|_m \|g_{\nu td}\|_n}, \tag{2.38}$$

whence

$$K_{\ell mnd} \geqslant \sup_{\mu,\nu>0} \mathscr{K}^{B_{st}}_{\ell mnd}(\mu, \nu). \tag{2.39}$$

The function $\mathscr{K}^{B_{st}}_{\ell mnd}$ can be computed from items (i) *and* (ii)*.*

Proof. See Section 6. □

"Fourier" lower bound on $K_{\ell mnd}$. As in [16], we use this term for the lower bound arising from the trial functions

$$f_{p\sigma d}(x) := e^{ipx_1} e^{-\sigma |x|^2/2} \quad \big(p \in [0, +\infty),\ \sigma \in (0, +\infty)\big). \tag{2.40}$$

The Sobolev norm of any order n of this function can be expressed using the modified Bessel function of the first kind I_ν, the Pochhammer symbol (1.8) and the double factorial (1.7).

Proposition 2.6. (i) *Let $m, p \in [0, +\infty)$, $\sigma \in (0, +\infty)$. Then*

$$\|f_{p\sigma d}\|^2_m = \frac{2\pi^{d/2}}{\sigma^{d/2+1} p^{d/2-1}} \int_0^{+\infty} d\rho\, \rho^{d/2} \big(1 + \rho^2\big)^m e^{-\frac{\rho^2+p^2}{\sigma}} I_{d/2-1}\bigg(\frac{2p}{\sigma}\rho\bigg) \tag{2.41}$$

if $p > 0$, and

$$\|f_{0\sigma d}\|^2_m = \frac{2\pi^{d/2}}{\Gamma(d/2)\sigma^d} \int_0^{+\infty} d\rho\, \rho^{d-1} \big(1 + \rho^2\big)^m e^{-\frac{\rho^2}{\sigma}} \tag{2.42}$$

(*this is the $p \to 0^+$ limit of* (2.41)*, since $I_{d/2-1}(w) \sim \frac{(w/2)^{d/2-1}}{\Gamma(d/2)}$ for $w \to 0^+$*).

In particular, for m integer,

$$\|f_{p\sigma d}\|^2_m = \pi^{d/2} \sum_{\ell=0}^{m} \sum_{j=0}^{\ell} \sum_{g=0}^{j} \binom{m}{\ell} \binom{\ell}{j} \binom{2j}{2g} \frac{(2g-1)!!}{2^g} \times (d/2 - 1/2)_{\ell-j}\, p^{2j-2g} \sigma^{\ell+g-j-d/2}. \tag{2.43}$$

(ii) *Let ℓ, m, n fulfill* (2.1)*. Then, for all $p, q \in [0, +\infty)$ and $\sigma, \tau \in (0, +\infty)$,*

$$K_{\ell mnd} \geqslant \mathscr{K}^{F}_{\ell mnd}(p, q, \sigma, \tau) := \frac{\|f_{p+q,\sigma+\tau,d}\|_\ell}{\|f_{p\sigma d}\|_m \|f_{q\tau d}\|_n}, \tag{2.44}$$

whence

$$K_{\ell mnd} \geqslant \sup_{p,q \geqslant 0,\, \sigma,\tau > 0} \mathscr{K}^F_{\ell mnd}(p,q,\sigma,\tau). \tag{2.45}$$

The function $\mathscr{K}^F_{\ell mnd}$ *can be computed from item* (i).

Proof. (i) See [16, Proposition 2.4].

(ii) Use Eq. (2.27) with $f = f_{p\sigma d}$ and $g = f_{q\tau d}$; then $fg = f_{p+q,\sigma+\tau,d}$ and we get Eq. (2.44). □

"S-constant" lower bound on $K_{\ell\ell nd}$. This lower bound holds for $K_{\ell mnd}$ in the special case $\ell = m$; it can be obtained from (2.27), substituting for f a family of approximants of the Dirac δ distribution, and then choosing for g the function g_{nd} of Eqs. (1.45) and (1.46). This bound already appeared in [14], analyzing an inequality strictly related to the case $\ell = m$ of (2.2). In the cited reference, for a number of reasons this was called the "ground level" lower bound; here, we prefer the denomination of "S-constant" lower bound to recall its relation with the Sobolev imbedding constant $S = S_{\infty nd}$ of Eq. (2.20).

Proposition 2.7. *Let*

$$0 \leqslant \ell \leqslant n, \quad n > \frac{d}{2}. \tag{2.46}$$

Then

$$K_{\ell\ell nd} \geqslant S_{\infty nd}. \tag{2.47}$$

Proof. As anticipated, it is essentially known from [14]; for completeness, a sketch of it is given in Section 7. □

The last statement, combined with the Hölder upper bound in Proposition 2.4, gives the sharp value of $K_{\ell\ell nd}$ in the trivial case $\ell = 0$.

Proposition 2.8. *Let* $n > d/2$; *then*

$$K_{00nd} = S_{\infty nd}. \tag{2.48}$$

Proof. We use Proposition 2.4 with $m = 0$ and n as above; in this case $R_{mnd} = R_{0nd} = \{2\}$, and $p = 2$ implies $p^* = +\infty$. Eq. (2.24) with $m = 0$, $p = 2$ gives

$$K_{00nd} \leqslant S_{20d} S_{\infty nd} = S_{\infty nd} \tag{2.49}$$

($S_{20d} = 1$, in agreement with (2.19). Note that (2.49) just rephrases the following obvious chain of inequalities, holding for all $f \in H^0(\mathbf{R}^d) = L^2(\mathbf{R}^d)$ and $g \in H^n(\mathbf{R}^d)$: $\|fg\|_0 = \|fg\|_{L^2} \leqslant \|f\|_{L^2}\|g\|_{L^\infty} \leqslant \|f\|_{L^2}(S_{\infty nd}\|g\|_n) = S_{\infty nd}\|f\|_0\|g\|_n$).

From (2.47) we have $K_{00nd} \geqslant S_{\infty nd}$ as well, so we get the thesis (2.48). □

In fact, the equality $K_{\ell\ell nd} = S_{\infty nd}$ holds as well in some cases with nonzero ℓ (e.g., for $d = 3$ and $\ell = 1, n = 2$: see Table 1 and Eqs. (B.27)–(B.29)).

3. On the explicit determination of upper and lower bounds for $K_{\ell mnd}$

Let us translate the results of the previous section into a scheme to get explicit upper and lower bounds $K^{\pm}_{\ell mnd}$ on $K_{\ell mnd}$, such that

$$K^{-}_{\ell mnd} \leqslant K_{\ell mnd} \leqslant K^{+}_{\ell mnd}.$$

At the end of the section, we present a table of such upper and lower bounds, for $d = 1$ or 3 and many values of ℓ, m, n. Before discussing the table, let us describe the general scheme to determine the upper and lower bounds.

On the computation of $K^{+}_{\ell mnd}$. One proceeds as follows.

(i) For any $\ell \geqslant 0$, one can use the $\mathscr{S}$-function upper bound provided by Proposition 2.2, Eq. (2.3), i.e., the number

$$K^{\mathscr{S}}_{\ell mnd} := \sqrt{\sup_{u\in[0,+\infty)} \mathscr{S}_{\ell mnd}(u)} \quad \text{(or an upper approximant for this).} \tag{3.1}$$

The function $\mathscr{S}_{\ell mnd}$ has the expression provided by Eqs. (2.4)–(2.14); depending on the case, its sup can be determined analytically or estimated numerically.

(ii) For $\ell = 0$, one can use as well the Hölder upper bound provided by Proposition 2.4, Eq. (2.25), i.e., the number

$$K^{\mathscr{H}}_{0mnd} := \inf_{p\in R_{mnd}} S_{pmd} S_{p^*nd} \quad \text{(or an upper approximant for this).} \tag{3.2}$$

Let us recall that $1/p + 1/p^* = 1/2$ and S_{pmd}, R_{pmd} are defined by Eqs. (2.18)–(2.21), (2.23); typically, the estimation of the inf over p is numerical.

(iii) We denote with $K^{+}_{\ell mnd}$ the best upper bound arising from (i) and (ii); so

$$K^{+}_{\ell mnd} := K^{\mathscr{S}}_{\ell mnd} \quad \text{if } \ell > 0, \qquad K^{+}_{0mnd} := \min\bigl(K^{\mathscr{S}}_{0mnd}, K^{\mathscr{H}}_{0mnd}\bigr). \tag{3.3}$$

On the computation of $K^{-}_{\ell mnd}$. One proceeds in this way (possibly using numerical methods to compute the quantities mentioned below).

(i) One chooses two values (s,t) fulfilling conditions (2.31); the choice $s = m$, $t = n$ is natural whenever possible. After fixing s, t one considers for $K_{\ell mnd}$ the Bessel lower bound suggested by Proposition 2.5, Eq. (2.39), i.e., the number

$$K^{B_{st}}_{\ell mnd} := \sup_{\mu,\nu>0} \mathscr{K}^{B_{st}}_{\ell mnd}(\mu,\nu) \quad \text{(or a lower approximant for this).} \tag{3.4}$$

The function $\mathscr{K}^{B_{st}}_{\ell mnd}$ is determined by Eqs. (2.30)–(2.38).

(ii) An alternative to the bound (3.4) is the Fourier lower bound suggested by Proposition 2.6, Eq. (2.45), i.e., the number

$$K^{F}_{\ell mnd} := \sup_{p,q\geqslant 0,\, \sigma,\tau>0} \mathscr{K}^{F}_{\ell mnd}(p,q,\sigma,\tau) \quad \text{(or a lower approximant for this).} \tag{3.5}$$

The function $\mathscr{K}^{F}_{\ell mnd}$ is determined by Eqs. (2.41)–(2.44).

Table 1
Table of the bounds $K^-_{\ell mnd} \leqslant K_{\ell mnd} \leqslant K^+_{\ell mnd}$ for $d = 1, 3$ and some values of ℓ, m, n (the notations $(F), (B_{st}), (S)$ indicate the type of the lower bound $K^-_{\ell mnd}$).

$d=1$					
ℓ	m	n	$K^+_{\ell mnd}$	$K^-_{\ell mnd}/K^+_{\ell mnd}$	
0	1	1	0.439	0.917	(B_{11})
0	1	2	0.383	0.987	(F)
0	1	10	0.274	0.997	(F)
1	1	2	0.562	0.916	(B_{12})
1	1	3	0.464	0.945	(B_{13})
1	1	10	0.310	0.984	$(B_{1,10})$
1	2	3	0.372	0.957	(B_{23})
2	2	3	0.564	0.842	(B_{23})
2	2	10	0.324	0.955	$(B_{2,10})$
2	3	3	0.419	0.907	(B_{33})
2	3	4	0.366	0.948	(B_{34})
2	3	10	0.284	0.971	$(B_{3,10})$
2	10	10	0.254	0.909	$(B_{10,10})$
4	5	6	0.417	0.878	(F)
10	10	11	1.238	0.817	(F)
10	11	11	0.969	0.825	(F)
10	11	12	0.804	0.845	(F)
10	11	20	0.391	0.906	(F)
10	20	20	0.214	0.888	(F)

$d=3$					
ℓ	m	n	$K^+_{\ell mnd}$	$K^-_{\ell mnd}/K^+_{\ell mnd}$	
0	1	1	0.135	0.842	(B_{22})
0	1	2	0.0694	0.918	(F)
0	1	10	0.0215	0.988	(F)
1	1	2	$1/2\sqrt{2\pi}$ (*)	1	(S)
1	1	3	0.101	0.987	(S)
1	1	10	0.0296	0.995	(S)
1	2	3	0.0581	0.865	(F)
2	2	3	0.115	0.916	(B_{23})
2	2	10	0.0302	0.981	$(B_{2,10})$
2	3	3	0.0646	0.901	(B_{33})
2	3	4	0.0482	0.916	(B_{34})
2	3	10	0.0237	0.909	$(B_{3,10})$
2	10	10	0.0167	0.754	(F)
4	5	6	0.0437	0.870	(F)
10	10	11	0.0990	0.798	(F)
10	11	11	0.0734	0.817	(F)
10	11	12	0.0583	0.833	(F)
10	11	20	0.0223	0.905	(F)
10	20	20	0.00978	0.974	(F)

* Note that $\frac{1}{2\sqrt{2\pi}} = 0.1994\ldots$. The equality $K^-_{1123}/K^+_{1123} = 1$ indicates that $\frac{1}{2\sqrt{2\pi}}$ is the sharp constant K_{1123}.

(iii) In the special case $\ell = m$, Proposition 2.7 also gives the S-constant lower bound

$$K_{\ell\ell nd} \geqslant S_{\infty nd},$$

with $S_{\infty nd}$ as in (2.20).

(iv) The best lower bound arising from (i), (ii) and (iii) is

$$K^-_{\ell mnd} := \max\left(K^{B_{st}}_{\ell mnd}, K^F_{\ell mnd}\right) \quad \text{if } \ell < m, \qquad K^-_{\ell\ell nd} := \max\left(K^{B_{st}}_{\ell\ell nd}, K^F_{\ell\ell nd}, S_{\infty nd}\right). \tag{3.6}$$

A table of upper and lower bounds. Table 1 considers the dimensions $d = 1, 3$ and a set of integer values for ℓ, m, n. For each one of these values an upper bound $K^+_{\ell mnd}$ and a lower bound $K^-_{\ell mnd}$ have been computed with the methods outlined above. Then, the values of $K^+_{\ell mnd}$ and of the ratio $K^-_{\ell mnd}/K^+_{\ell mnd}$ have been reported in the table: giving the above ratio, rather than the lower bound, is more convenient to appreciate how narrow is the uncertainty on $K_{\ell mnd}$.

In all cases considered in the table $\mathscr{S}_{\ell mnd}$, $\mathscr{K}^{B_{st}}_{\ell mnd}$ and $\mathscr{K}^F_{\ell mnd}$ are elementary functions, but often they have lengthy expressions; typically, their sups or infs have been evaluated numerically. The long expressions for the cited functions have been obtained implementing the general formulas of Section 2 on MATHEMATICA, in the symbolic mode; the same package, with its standard optimization algorithms, has been employed to compute numerically the necessary sups and infs.

In the cases $\ell = 0$ of the table, the minimum (3.3) giving K^+_{0mnd} equals $K^{\mathscr{H}}_{0mnd}$.

Depending on the case, the lower bound $K^-_{\ell mnd}$ in (3.6) can either be a Bessel bound $K^{B_{st}}_{\ell mnd}$, a Fourier bound $K^F_{\ell mnd}$ or an S-constant bound $S_{\infty nd}$; to distinguish these situations we have placed after the value of $K^-_{\ell mnd}/K^+_{\ell mnd}$ the symbols (B_{st}), (F) or (S), respectively.

Here we present the table of upper and lower bounds; in Appendix B we give some examples of the calculations from which the table originated, reporting all the necessary details.

4. Asymptotics for the upper and lower bounds on $K_{\ell mnd}$

As reviewed in the introduction, in our previous work on the constant $K_{\ell\ell\ell d} \equiv K_{\ell d}$ we have analyzed the $\ell \to +\infty$ asymptotics of some upper and lower bounds for this constant, the conclusion being (1.4).

Now, we are in condition to analyze more general limit cases; here we discuss the behavior of $K_{\ell mnd}$ when

$$m = b\ell, \qquad n = c\ell \quad (1 \leqslant b \leqslant c), \quad \ell \to +\infty. \tag{4.1}$$

We note that conditions (2.1) on ℓ, $m = b\ell$, $n = c\ell$ and d are fulfilled if

$$1 \leqslant b \leqslant c, \quad \ell > \frac{d}{2(b+c-1)}. \tag{4.2}$$

Let us first analyze the asymptotics of an upper bound for $K_{\ell mnd}$. Our starting point is the inequality

$$K_{\ell mnd} \leqslant K^{\mathscr{S}}_{\ell mnd} := \sqrt{\sup_{u\in[0,+\infty)} \mathscr{S}_{\ell mnd}(u)}, \tag{4.3}$$

with $\mathscr{S}_{\ell mnd}$ as in Eq. (2.4), to be used with $m = b\ell$ and $n = c\ell$. We note that Eqs. (2.4) and (2.5) give

$$\mathscr{S}_{\ell,b\ell,c\ell,d}(u) = \frac{\Gamma((b+c)\ell - d/2)}{(4\pi)^{d/2}\Gamma((b+c)\ell)} \Sigma_{bcd\ell}(u), \tag{4.4}$$

$$\Sigma_{bcd\ell} : [0,+\infty) \to (0,+\infty),$$

$$u \mapsto \Sigma_{bcd\ell}(u) := (1+4u)^{\ell}\, {}_3F_2\left((b+c)\ell - \frac{d}{2}, b\ell, c\ell; \frac{(b+c)\ell}{2}, \frac{(b+c)\ell+1}{2}; -u\right). \tag{4.5}$$

Our subsequent analysis rests on the condition introduced hereafter.

Definition 4.1. Let $1 \leqslant b \leqslant c$, and $d \in \mathbf{N}_0$. We say that *condition* $\mathfrak{S}_{bcd}$ *holds* if

$$\sup_{u\in[0,+\infty)} \Sigma_{bcd\ell}(u) = 1 + O(1/\ell) \quad \text{for } \ell \to +\infty. \tag{4.6}$$

Remarks 4.2. (i) In any case, $\Sigma_{bcd\ell}(0) = 1$. So, the above condition means that $\sup_u \Sigma_{bcd\ell}$ is close to the value of the function at $u = 0$.

(ii) Condition $\mathfrak{S}_{11d}$ *does not* hold for any $d \in \mathbf{N}_0$. In fact, with the present notations, Proposition 2.2 of [16] gives $\sup_{u\in[0,+\infty)} \Sigma_{11d\ell}(u) = \Sigma_{11d\ell}(1/2)[1 + O(1/\ell)] = 3^{d/2+1/2}2^{-d/2}(4/3)^{\ell}[1 + O(1/\ell)]$ for $\ell \to +\infty$.

On the other hand, this negative result is not important for our purposes: in fact the case $b = c = 1$, i.e., $\ell = m = n$, is just the one analyzed by different means in [16], and summarized here via Eq. (1.4).

Hereafter we consider a case where $\mathfrak{S}_{bcd}$ can be proved, and another one where it can be reasonably conjectured.

Proposition 4.3. *Condition* $\mathfrak{S}_{22d}$ *holds for each* $d \in \mathbf{N}_0$.

Proof. See Section 8. □

Remark 4.4. The above result is sufficient for our purposes, but there is evidence for a slightly stronger statement: $\sup_{u\geqslant 0}\Sigma_{22d\ell}$ is attained at a point $u=u_{22d\ell}\neq 0$ that, for $\ell\to+\infty$, converges to zero in such a way to fulfill condition (4.6). We return to this point in the forthcoming Remark 8.2.

Let us pass from the case $b=c=2$ to $b=2, c=3$; for the latter we have found numerical evidence (but no analytic proof) for the following conjecture.

Conjecture 4.5. *For each $d\in\mathbf{N}_0$ there is a real number $\ell_d>d/8$ such that, for all $\ell\geqslant\ell_d$, the function $\Sigma_{23d\ell}$ is strictly decreasing on $[0,+\infty)$. So*

$$\sup_{u\in[0,+\infty)}\Sigma_{23d\ell}(u)=\Sigma_{23d\ell}(0)=1\quad\textit{for each }\ell\geqslant\ell_d \tag{4.7}$$

(*which implies condition $\mathfrak{S}_{23d}$, in a strong version with no term $O(1/\ell)$ in Eq.* (4.6)).

In the above, the condition $\ell_d>d/8$ reflects the inequality on ℓ in Eq. (4.2), for $b=2$ and $c=3$. Conjecture 4.5 is probably related to some inequalities for the ${}_{q+1}F_q$ functions, conjectured in [8].

Proposition 4.6. *Suppose condition $\mathfrak{S}_{bcd}$ to hold for some fixed b,c,d ($1\leqslant b\leqslant c$, $d\in\mathbf{N}_0$). Then, the upper bound $K^{\mathscr{S}}_{\ell,b\ell,c\ell,d}$ on $K_{\ell,b\ell,c\ell,d}$ has the asymptotics*

$$K^{\mathscr{S}}_{\ell,b\ell,c\ell,d}=\frac{1+O(1/\ell)}{[4(b+c)\pi\ell]^{d/4}}\quad\textit{for }\ell\to+\infty. \tag{4.8}$$

Proof. Let $\ell\to+\infty$. Eqs. (4.3)–(4.6) give

$$K^{\mathscr{S}}_{\ell,b\ell,c\ell,d}=\sqrt{\frac{\Gamma((b+c)\ell-d/2)}{\Gamma((b+c)\ell)}}\,\frac{1+O(1/\ell)}{(4\pi)^{d/4}}. \tag{4.9}$$

Now, the thesis follows using the relation

$$\frac{\Gamma((b+c)\ell-d/2)}{\Gamma((b+c)\ell)}=\frac{1+O(1/\ell)}{[(b+c)\ell]^{d/2}}, \tag{4.10}$$

which is a consequence of Eq. (1.15). □

Let us pass to the asymptotics for a suitable lower bound on $K_{\ell,b\ell,c\ell,d}$. We recall that, for any ℓ,m,n, we have the Fourier lower bound (2.44); let us use this with $p=q=0$. So, for all $\sigma,\tau\in(0,+\infty)$,

$$K_{\ell mnd}\geqslant\mathscr{K}^F_{\ell mnd}(\sigma,\tau):=\frac{\|f_{\sigma+\tau,d}\|_\ell}{\|f_{\sigma d}\|_m\|f_{\tau d}\|_n}; \tag{4.11}$$

here $f_{\sigma d}:=f_{p=0,\sigma,d}$, i.e.,

$$f_{\sigma d}:\mathbf{R}^d\to\mathbf{R},\quad x\mapsto f_{\sigma d}(x):=e^{-\sigma|x|^2/2}\quad\big(\sigma\in(0,+\infty)\big). \tag{4.12}$$

Our main result in this framework is the following.

Proposition 4.7. *Let* $1 \leqslant b \leqslant c$, $d \in \mathbf{N}_0$, *and*

$$\Delta_{bc} := \{(\xi, \eta) \in (0, 1/b) \times (0, 1/c) \mid \xi + \eta < 1\}. \tag{4.13}$$

Then, for fixed $(\xi, \eta) \in \Delta_{bc}$ *and* $\ell \to +\infty$,

$$\mathscr{K}^{F}_{\ell,b\ell,c\ell,d}\left(\frac{\xi}{\ell}, \frac{\eta}{\ell}\right) = \frac{1 + O(1/\ell)}{[D_{bc}(\xi, \eta)\pi \ell]^{d/4}}, \quad D_{bc}(\xi, \eta) := \frac{(1 - \xi - \eta)(\xi + \eta)}{\xi\eta(1 - b\xi)(1 - c\eta)}. \tag{4.14}$$

Proof. See Section 8. $\square$

For given b, c one uses Eq. (4.14) choosing $(\xi, \eta) \in \Delta_{bc}$ so as to minimize D_{bc} (or to go as close as possible to the minimum point of this function); this choice gives the best lower bound of the type (4.14), in the limit $\ell \to +\infty$.

Let us write down two corollaries of Propositions 4.6 and 4.7, for the cases $b = c = 2$ and $b = 2$, $c = 3$, respectively.

Corollary 4.8. *For any* $d \in \mathbf{N}_0$, *the following hold*:

(i) *The upper bound* $K^{\mathscr{S}}_{\ell,2\ell,2\ell,d}$ *is such that*

$$K^{\mathscr{S}}_{\ell,2\ell,2\ell,d} = \frac{1 + O(1/\ell)}{(16\pi \ell)^{d/4}} \quad \textit{for } \ell \to +\infty. \tag{4.15}$$

(ii) *The function* $D_{22} : \Delta_{22} \to (0, +\infty)$ *from Proposition* 4.7 *attains its minimum at* $\xi = \eta = 1/4$. *It is* $D_{22}(1/4, 1/4) = 16$; *so, the corresponding lower bound* $\mathscr{K}^{F}_{\ell,2\ell,2\ell,d}(1/(4\ell), 1/(4\ell))$ *is such that*

$$\mathscr{K}^{F}_{\ell,2\ell,2\ell,d}\left(\frac{1}{4\ell}, \frac{1}{4\ell}\right) = \frac{1 + O(1/\ell)}{(16\pi \ell)^{d/4}} \quad \textit{for } \ell \to +\infty. \tag{4.16}$$

(iii) *As a consequence of* (i) *and* (ii), *one has*

$$K_{\ell,2\ell,2\ell,d} = \frac{1 + O(1/\ell)}{(16\pi \ell)^{d/4}} \quad \textit{for } \ell \to +\infty. \tag{4.17}$$

Proof. (i) Use Proposition 4.6 with $b = c = 2$ (recalling that condition $\mathfrak{S}_{22}$ holds, by Proposition 4.3).
(ii) Elementary.
(iii) The thesis follows from $\mathscr{K}^{F}_{\ell,2\ell,2\ell,d}(\frac{1}{4\ell}, \frac{1}{4\ell}) \leqslant K_{\ell,2\ell,2\ell,d} \leqslant K^{\mathscr{S}}_{\ell,2\ell,2\ell,d}$. $\square$

Corollary 4.9. *For* $d \in \mathbf{N}_0$, *we have the following*:

(i) *If* $\mathfrak{S}_{23d}$ *holds, the upper bound* $K^{\mathscr{S}}_{\ell,2\ell,3\ell,d}$ *is such that*

$$K^{\mathscr{S}}_{\ell,2\ell,3\ell,d} = \frac{1 + O(1/\ell)}{(20\pi \ell)^{d/4}} \quad \textit{for } \ell \to +\infty. \tag{4.18}$$

(ii) *Consider the function* $D_{23} : \Delta_{23} \to (0, +\infty)$ *from Proposition* 4.7, *and evaluate it at* $(\xi, \eta) := (1/5, 1/7)$ (*which is close to its minimum point*). *It is* $D_{23}(1/5, 1/7) = 23$; *so the corresponding lower bound* $\mathscr{K}^F_{\ell,2\ell,3\ell,d}(1/(5\ell), 1/(7\ell))$ *is such that*

$$\mathscr{K}^F_{\ell,2\ell,3\ell,d}\left(\frac{1}{5\ell}, \frac{1}{7\ell}\right) = \frac{1 + O(1/\ell)}{(23\pi\ell)^{d/4}} \quad \textit{for } \ell \to +\infty. \tag{4.19}$$

(iii) *Summing up,* (i) *and* (ii) *give*

$$\frac{1 + O(1/\ell)}{(23\pi\ell)^{d/4}} \leqslant K_{\ell,2\ell,2\ell,d} \underset{(\mathfrak{S}_{23d})}{\leqslant} \frac{1 + O(1/\ell)}{(20\pi\ell)^{d/4}} \quad \textit{for } \ell \to +\infty, \tag{4.20}$$

where $\underset{(\mathfrak{S}_{23d})}{\leqslant}$ *means that the indicated relation is true if condition* $\mathfrak{S}_{23d}$ *holds.*

Proof. (i) Use Proposition 4.6 with $b = 2$, $c = 3$. (ii) Elementary. (iii) Obvious. □

5. Proof of Proposition 2.2

Here and in the rest of the paper, we work in a fixed dimension $d \in \mathbf{N}_0$. The proof of the cited proposition is preceded by some lemmas. The method is similar to the one of [16], but technically more difficult; again, the basic idea is to work with the Fourier transform $\mathcal{F}$, that sends the pointwise product of functions into the convolution.

Let us write $F * G$ for the convolution of two complex functions F, G on $\mathbf{R}^d$, given by

$$(F * G)(k) := \int_{\mathbf{R}^d} dh\, F(k - h)G(h). \tag{5.1}$$

We have

$$\mathcal{F}(fg) = \frac{1}{(2\pi)^{d/2}} \mathcal{F}f * \mathcal{F}g \tag{5.2}$$

for all sufficiently regular functions f and g on $\mathbf{R}^d$ (and, in particular, for functions to which we will apply (5.2) in the rest of the section).

Let us recall the definition (1.44) $G_{td}(k) := 1/(1 + |k|^2)^t$ for all $t \in \mathbf{R}$ and $k \in \mathbf{R}^d$, to which we will refer systematically in the sequel. The forthcoming lemmas consider pairs m, n or triples ℓ, m, n of real numbers.

Lemma 5.1. *Let* $m + n > d/2$. *Then, the integral defining the convolution* $(G_{md} * G_{nd})(k)$ *is convergent, for all* $k \in \mathbf{R}^d$.

Proof. For an integral $\int_{\mathbf{R}^d} F(h)\, dh$ to be convergent, it suffices that F be continuous and that, for $h \to \infty$, $F(h) = O(1/|h|^\eta)$ with $\eta > d$. For any $k \in \mathbf{R}^d$, the convolution integral

$$(G_{md} * G_{nd})(k) = \int_{\mathbf{R}^d} \frac{dh}{(1 + |k - h|^2)^m (1 + |h|^2)^n} \tag{5.3}$$

fulfills these conditions with $\eta = 2(m + n)$. □

Lemma 5.2. *Let ℓ, m, n fulfill* (2.1)*. Then*

$$K_{\ell mnd} \leqslant \sqrt{\sup_{k \in \mathbf{R}^d} \mathcal{S}_{\ell mnd}(k)}, \tag{5.4}$$

$$\mathcal{S}_{\ell mnd}(k) := \frac{(1+|k|^2)^\ell}{(2\pi)^d}(G_{md} * G_{nd})(k). \tag{5.5}$$

Proof. Consider any two functions $f \in H^m(\mathbf{R}^d, \mathbf{C})$, $g \in H^n(\mathbf{R}^d, \mathbf{C})$. Then

$$\|fg\|_\ell^2 = \int_{\mathbf{R}^d} dk\big(1+k^2\big)^\ell \big|\mathcal{F}(fg)(k)\big|^2 = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} dk\big(1+k^2\big)^\ell \big|(\mathcal{F}f * \mathcal{F}g)(k)\big|^2. \tag{5.6}$$

Explicitating the convolution we find

$$\begin{aligned}(\mathcal{F}f * \mathcal{F}g)(k) &= \int_{\mathbf{R}^d} dh\, \mathcal{F}f(k-h)\mathcal{F}g(h) \\ &= \int_{\mathbf{R}^d} dh\, \frac{1}{\sqrt{1+|k-h|^2}^{\,m}\sqrt{1+|h|^2}^{\,n}} \\ &\quad \times \Big(\sqrt{1+|k-h|^2}^{\,m}\mathcal{F}f(k-h)\sqrt{1+|h|^2}^{\,n}\mathcal{F}g(h)\Big),\end{aligned} \tag{5.7}$$

and Hölder's inequality $|\int dh\, U(h)V(h)|^2 \leqslant \int dh\, |U(h)|^2 \int dh\, |V(h)|^2$ gives

$$\begin{aligned}&\big|(\mathcal{F}f * \mathcal{F}g)(k)\big|^2 \leqslant C(k)P(k), \\ &C(k) := \int_{\mathbf{R}^d} \frac{dh}{(1+|k-h|^2)^m(1+|h|^2)^n} = (G_{md} * G_{nd})(k), \\ &P(k) := \int_{\mathbf{R}^d} dh\big(1+|k-h|^2\big)^m\big|\mathcal{F}f(k-h)\big|^2\big(1+|h|^2\big)^n\big|\mathcal{F}g(h)\big|^2.\end{aligned} \tag{5.8}$$

Inserting (5.8) into Eq. (5.6) we get

$$\begin{aligned}\|fg\|_\ell^2 &\leqslant \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} dk\big(1+|k|^2\big)^\ell C(k)P(k) \\ &\leqslant \Big(\sup_{k\in\mathbf{R}^d}\frac{(1+|k|^2)^\ell}{(2\pi)^d}C(k)\Big)\int_{\mathbf{R}^d} dk\, P(k) = \Big(\sup_{k\in\mathbf{R}^d}\mathcal{S}_{\ell mnd}(k)\Big)\int_{\mathbf{R}^d} dk\, P(k).\end{aligned} \tag{5.9}$$

But

$$\int_{\mathbf{R}^d} dk\, P(k) = \int_{\mathbf{R}^d} dk\,\big(1+|k|^2\big)^m\big|\mathcal{F}f(k)\big|^2\int_{\mathbf{R}^d} dh\,\big(1+|h|^2\big)^n\big|\mathcal{F}g(h)\big|^2 = \|f\|_m^2\|g\|_n^2,$$

so we are led to the thesis. □

Lemma 5.3. *Let $m, n \geqslant 0$, $m+n > d/2$. Then, for all $k \in \mathbf{R}^d$,*

$$(G_{md} * G_{nd})(k) = \pi^{d/2} \frac{\Gamma(m+n-d/2)}{\Gamma(m+n)} F_{mnd}\left(\frac{|k|^2}{4}\right), \tag{5.10}$$

where F_{mnd} is the hypergeometric function (of the ${}_3F_2$ type) in Eq. (2.5) of Proposition 2.2.

Proof. Both sides of (5.10) are symmetric in m, n, so we can restrict the attention to the case $m \leqslant n$ and write our basic assumptions as

$$0 \leqslant m \leqslant n, \quad m+n > \frac{d}{2}. \tag{5.11}$$

Conditions (5.11) on m, n are equivalent to

$$\frac{d}{4} < n, \quad m \in M_{nd}, \quad M_{nd} := [0, n] \cap (d/2 - n, +\infty). \tag{5.12}$$

Let us fix $k \in \mathbf{R}^d$. We claim that it is sufficient to prove the thesis (5.10) under even more restrictive conditions than (5.12), namely, for

$$\frac{d}{4} < m \leqslant n. \tag{5.13}$$

In fact, for fixed ($k \in \mathbf{R}^d$ and) $n > d/4$:

(i) both sides of Eq. (5.10), viewed as functions of m, are analytic in an open neighborhood on M_{nd}, namely, the interval $(d/2 - n, +\infty)$. This is made evident by the expression (5.3) for the convolution integral $(G_{md} * G_{nd})(k)$ and by the considerations about ${}_{q+1}F_q$ following Eq. (1.24), here applied to $F_{mnd}(|k|^2/4) = {}_3F_2(m+n-d/2, m, n; (m+n)/2, (m+n+1)/2, -|k|^2/4)$.[1]
(ii) By the principle of analytic continuation, if the two sides of (5.10) are equal for $m \in (d/4, n]$, they are equal as well for m in M_{nd}.

The rest of the proof is devoted to establishing (5.10) for m, n as in (5.13).

Under these conditions we can represent G_{td} as the Fourier transform of the function g_{td} (Eqs. (1.45) and (1.46)), both for $t = n$ and for $t = m$. From here and (5.2),

$$(G_{md} * G_{nd})(k) = (2\pi)^{d/2} \mathcal{F}(g_{md} g_{nd})(k). \tag{5.14}$$

The product $g_{md} g_{nd}$ is a radially symmetric function, whose explicit expression in terms of Macdonald functions follows from (1.46). So, $\mathcal{F}(g_{md} g_{nd})$ can be computed using the formula (1.35) for radially symmetric Fourier transforms, and the conclusion is

$$(G_{md} * G_{nd})(k) = \frac{(2\pi)^{d/2}}{2^{m+n-2}\Gamma(n)\Gamma(m)|k|^{d/2-1}} \times \int_0^{+\infty} dr\, r^{m+n-d/2} J_{d/2-1}\big(|k|r\big) K_{m-d/2}(r) K_{n-d/2}(r); \tag{5.15}$$

[1] The analyticity result for ${}_{q+1}F_q$ stated after the integral representation (1.24) ensures the following in the present case, for fixed $n > d/4$ and $u \in [0, +\infty)$: the function $m \mapsto {}_3F_2(m+n-d/2, m, n; m/2+n/2, m/2+n/2+1/2, -u)$ is analytic where m fulfills the condition $m+n-d/2 > 0$, i.e., for $m \in (d/2-n, +\infty)$.

the above integral is computed via (1.32), and in this way one gets the thesis (5.10). (Final remark: some of our last manipulations seem to exclude the point $k = 0$, see e.g. the denominator in Eq. (5.15); however, Eq. (5.10) holds here as well, by continuity.) □

Lemma 5.4. *Let ℓ, m, n fulfill* (2.1)*. Then, for all $k \in \mathbf{R}^d$,*

$$\mathcal{S}_{\ell mnd}(k) = \mathscr{S}_{\ell mnd}\big(|k|^2/4\big), \tag{5.16}$$

where $\mathscr{S}_{\ell mnd}$ is the function in Eq. (2.4) *of Proposition* 2.2.

Proof. This follows immediately from the definition (5.5) $\mathcal{S}_{\ell mnd}(k) := \frac{(1+|k|^2)^\ell}{(2\pi)^d}(G_{md} * G_{nd})(k)$, from Eq. (5.10) of the previous lemma and from the definition (2.4) of $\mathscr{S}_{\ell mnd}$. □

We are finally ready to derive the main result of the section, i.e., to prove Proposition 2.2.

Proof of Proposition 2.2, item (i). Again, ℓ, m, n are assumed to fulfill (2.1). Lemmas 5.2 and 5.4 give immediately the bound (2.3) for $K_{\ell mnd}$, with $\mathscr{S}_{\ell mnd}$ as in Eq. (2.4); in the sequel we frequently mention the hypergeometric function F_{mnd} appearing in Eqs. (2.4) and (2.5), recalling again that this is of the ${}_3F_2$ type.

In the special case $m = n$, the expression (2.6) of F_{mnd} as a ${}_2F_1$ function follows immediately from (1.25). Eq. (2.7) for the "trivial" case $m = 0$ arises noting that $F_{0nd}(u) = {}_3F_2(n - d/2, 0, n; n/2, n/2 + 1/2, -u) = 1$ by (1.22).

Let us prove the properties of $\mathscr{S}_{\ell mnd}$ mentioned in item (i), for arbitrary ℓ, m, n, d.

First of all, the statement $\mathscr{S}_{\ell mnd}(u) \in (0, +\infty)$ for all $u \in [0, +\infty)$ follows immediately from the relation (5.16) between this function and $\mathcal{S}_{\ell mnd}$, which is positive due to the definition (5.5). Any hypergeometric function ${}_pF_q$ takes the value 1 at the origin; so, $\mathscr{S}_{\ell mnd}(0)$ has the expression (2.8). To conclude, we must prove the asymptotics (2.9) for $\mathscr{S}_{\ell mnd}(u)$ as $u \to +\infty$; this will give the result (2.10) for $\mathscr{S}_{\ell mnd}(+\infty)$, also implying the boundedness of $\mathscr{S}_{\ell mnd}$ on $[0, +\infty)$.

To derive (2.9), we first consider the case $m < n$ and apply to $F_{mnd}(u)$ the general asymptotics (1.31) (with $\alpha = m$, $\beta = n$, $\gamma = m + n - d/2$); with the obvious relation $(1 + 4u)^\ell \sim (4u)^\ell$, this gives

$$\mathscr{S}_{\ell mnd}(u) \sim \frac{4^\ell}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \frac{\Gamma_{mn}}{u^{m-\ell}} \quad \text{for } u \to +\infty,$$
$$\Gamma_{mn} := \frac{\Gamma(\frac{m}{2} + \frac{n}{2})\Gamma(\frac{m}{2} + \frac{n}{2} + \frac{1}{2})}{\Gamma(m+n)} \frac{\Gamma(n-m)}{\Gamma(\frac{n}{2} - \frac{m}{2})\Gamma(\frac{n}{2} - \frac{m}{2} + \frac{1}{2})}. \tag{5.17}$$

On the other hand, expressing $\Gamma(n \pm m)$ via the duplication formula (1.13) we see that

$$\Gamma_{mn} = \frac{1}{4^m} \quad \text{for all } n; \tag{5.18}$$

Eqs. (5.18) and (5.17) give the thesis (2.9), with the previous assumption $m < n$. To conclude, we must derive (2.9) in the special case $m = n$, where F_{mnd} collapses into a ${}_2F_1$ function due to (2.6); this case is worked out similarly to the previous one, using the asymptotics (1.30) (and again, the duplication formula for Γ). □

Proof of Proposition 2.2, item (ii). Our aim is to derive the series expansions for F_{mnd} in the cited item of the proposition, and to show that they are just finite sums with the special assumptions on m, n, d indicated therein.

First of all we note that, for $u \in [0, +\infty)$,

$$F_{mnd}(u) = \sum_{i=0}^{+\infty} \frac{(m+n-\frac{d}{2})_i (m)_i (\frac{m-n+1}{2})_i}{(\frac{m+n}{2})_i (\frac{m+n+1}{2})_i} \frac{u^i}{i!} {}_2F_1\left(m+n-\frac{d}{2}+i, m+i; \frac{m+n}{2}+i; -u\right)$$
$$= \sum_{i=0}^{+\infty} \frac{(m+n-\frac{d}{2})_i (m)_i (\frac{m-n}{2})_i}{(\frac{m+n+1}{2})_i (\frac{m+n}{2})_i} \frac{u^i}{i!} {}_2F_1\left(m+n-\frac{d}{2}+i, m+i; \frac{m+n+1}{2}+i; -u\right). \tag{5.19}$$

In the above, the first equality follows directly from the definition (2.5) and from the expansion (1.29); the second equality follows writing $F_{mnd}(u) = {}_3F_2(m+n-\frac{d}{2}, m, n; \frac{m+n+1}{2}, \frac{m+n}{2}; -u)$, and then using again Eq. (1.29). On the other hand,

$${}_2F_1\left(m+n-\frac{d}{2}+i, m+i; \frac{m+n}{2}+i; -u\right)$$
$$= \frac{{}_2F_1(\frac{d-m-n}{2}, \frac{n-m}{2}; \frac{m+n}{2}+i; -u)}{(1+u)^{\frac{3m+n-d}{2}+i}}$$
$$= \frac{1}{(1+u)^{\frac{3m+n-d}{2}+i}} \sum_{j=0}^{+\infty} \frac{(\frac{d-m-n}{2})_j (\frac{n-m}{2})_j}{(\frac{m+n}{2}+i)_j} \frac{(-u)^j}{j!}; \tag{5.20}$$

the first equality above follows from the Kummer transformation (1.26), the second one reflects the standard power series expansion (1.18) for ${}_2F_1$. The latter expansion holds if $u \in [0, 1)$, or $u \in [0, +\infty)$ and the series over j is a finite sum; these are just the conditions in the proposition under proof.

Inserting the expansion (5.20) into the first equality (5.19), one gets (2.11).

For similar reasons, we can write

$${}_2F_1\left(m+n-\frac{d}{2}+i, m+i; \frac{m+n+1}{2}+i; -u\right)$$
$$= \frac{{}_2F_1(\frac{d+1-m-n}{2}, \frac{n+1-m}{2}; \frac{m+n+1}{2}+i; -u)}{(1+u)^{\frac{3m+n-d-1}{2}+i}}$$
$$= \frac{1}{(1+u)^{\frac{3m+n-d-1}{2}+i}} \sum_{j=0}^{+\infty} \frac{(\frac{d+1-m-n}{2})_j (\frac{n+1-m}{2})_j}{(\frac{m+n+1}{2}+i)_j} \frac{(-u)^j}{j!} \tag{5.21}$$

(again when $u \in [0, 1)$, or $u \in [0, +\infty)$ and the series over j is a finite sum). Inserting this result into the second equality (5.19), one gets (2.12).

We finally come to statements (2.13)–(2.14), giving conditions for the series over j, i in (2.11) or (2.12) to become finite sums; as an example, we account for the first of such statements.

The series over j in (2.11) contains the Pochhammer symbol $(\frac{d-m-n}{2})_j$; on the other hand, the assumption in the first line of (2.13) is equivalent to

$$\frac{d-m-n}{2} = -h, \quad h \in \mathbf{N}. \tag{5.22}$$

From $h \in \mathbf{N}$ we infer $(-h)_j = 0$ for $j > h$, so

$$\sum_{j=0}^{+\infty} \to \sum_{j=0}^{h} = \sum_{j=0}^{\frac{m+n-d}{2}} \quad \text{in (2.11).} \tag{5.23}$$

The other statements in (2.13)–(2.14) are proved analyzing: the term $(\frac{m-n+1}{2})_i$ in (2.11); the term $(\frac{d+1-m-n}{2})_j$ in (2.12); the term $(\frac{m-n}{2})_i$ in (2.12). □

6. Proof of Proposition 2.5

Hereafter we prove items (i) and (ii) of the cited proposition (after this, item (iii) will be obvious).

(i) We must show that $g_{\nu td}$ belongs to $H^n(\mathbf{R}^d, \mathbf{C})$, and justify the expression (2.30) for its H^n norm. The relation $g_{\nu td} \in H^n(\mathbf{R}^d, \mathbf{C})$ follows from the finiteness of the integrals appearing below; the norm of this function is given by

$$\begin{aligned} \|g_{\nu td}\|_n^2 &= \int_{\mathbf{R}^d} dk \left(1+|k|^2\right)^n \left|\mathcal{F}g_{\nu td}(k)\right|^2 = \frac{1}{\nu^{2d}} \int_{\mathbf{R}^d} dk \, \frac{(1+|k|^2)^n}{(1+|k|^2/\nu^2)^{2t}} \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)\nu^{2d}} \int_0^{+\infty} d\rho \, \rho^{d-1} \frac{(1+\rho^2)^n}{(1+\rho^2/\nu^2)^{2t}} = \frac{\pi^{d/2}}{\Gamma(d/2)\nu^d} \int_0^{+\infty} du \, u^{d/2-1} \frac{(1+\nu^2 u)^n}{(1+u)^{2t}}. \end{aligned} \tag{6.1}$$

In the last two passages we have used Eq. (1.10) for the integral of a radially symmetric function, depending only on $\rho := |k|$, and then we have changed the variable ρ to $u = \rho^2/\nu^2$.

Let us fix the attention to the integral over u (clearly convergent, due to the assumption $t > n/2 + d/4$ in the statement under proof); this integral is computed via the identity (1.28), and one gets the thesis (2.30).

(ii) In the proof of Lemma 5.4, we have derived Eq. (5.15) for a Fourier transform of the type $\mathcal{F}(g_{md}g_{nd})$. With similar manipulations, in this case we get

$$\begin{aligned} \mathcal{F}(g_{\mu sd}g_{\nu td})(k) = {} & \frac{\mu^{s-d/2}\nu^{t-d/2}}{2^{s+t-2}\Gamma(s)\Gamma(t)|k|^{d/2-1}} \\ & \times \int_0^{+\infty} dr \, r^{s+t-d/2} J_{d/2-1}\big(|k|r\big) K_{s-d/2}(\mu r) K_{t-d/2}(\nu r), \end{aligned} \tag{6.2}$$

and a coordinate change $r \to r/|k|$ gives

$$\mathcal{F}(g_{\mu sd}g_{\nu td})(k) = G_{std}\big(\mu, \nu; |k|^2/4\big), \tag{6.3}$$

with G_{std} as in (2.33). This implies

$$\begin{aligned} \|g_{\mu sd}g_{\nu td}\|_\ell^2 &= \int_{\mathbf{R}^d} dk \left(1+|k|^2\right)^\ell \left|\mathcal{F}(g_{\mu sd}g_{\nu td})(k)\right|^2 \\ &= \int_{\mathbf{R}^d} dk \left(1+|k|^2\right)^\ell G_{std}^2\big(\mu, \nu; |k|^2/4\big). \end{aligned} \tag{6.4}$$

On the other hand, for radial integrals we have $dk = 2\pi^{d/2}|k|^{d-1}\,d|k|/\,\Gamma(d/2)$, and putting $|k| = 2\sqrt{u}$ we get the expression (2.32) for $\|g_{\mu sd}\,g_{\nu td}\|^2_\ell$.

Finally, let us consider the case $s - \frac{d}{2}, t - \frac{d}{2} \in \mathbf{N} + \frac{1}{2}$, $\ell \in \mathbf{N}$, and show that Eqs. (2.32) and (2.33) yield Eq. (2.35). To this purpose, we first compute the function $G_{std}(\mu, \nu; u)$ in (2.33); in this case Eq. (1.16) for the Macdonald functions gives

$$G_{std}(\mu, \nu; u) = \frac{\pi}{2^{2s+2t-2}\Gamma(s)\Gamma(t)} \times \sum_{i=0}^{s-\frac{d}{2}-\frac{1}{2}} \sum_{j=0}^{t-\frac{d}{2}-\frac{1}{2}} \frac{(2s-i-d-1)!(2t-j-d-1)!\mu^i\nu^j}{i!j!(s-i-\frac{d}{2}-\frac{1}{2})!(t-j-\frac{d}{2}-\frac{1}{2})!u^{i/2+j/2+d/2}} \times \int_0^{+\infty} dr\, r^{i+j+d/2} J_{d/2-1}(r) e^{-\frac{(\mu+\nu)r}{2\sqrt{u}}}. \tag{6.5}$$

On the other hand, for any $\sigma \in (0, +\infty)$,

$$\begin{aligned}
&\int_0^{+\infty} dr\, r^{i+j+d/2} J_{d/2-1}(r) e^{-r/\sigma} \\
&= \frac{(i+j+d-1)!\sigma^{i+j+d}}{2^{d/2-1}\Gamma(d/2)}\, {}_2F_1\left(\frac{i+j+d}{2}, \frac{i+j+d+1}{2}; \frac{d}{2}; -\sigma^2\right) \\
&= \frac{(i+j+d-1)!\sigma^{i+j+d}}{2^{d/2-1}\Gamma(d/2)(1+\sigma^2)^{i+j+d/2+1/2}}\, {}_2F_1\left(-\frac{i+j}{2}, -\frac{i+j+1}{2}; \frac{d}{2}; -\sigma^2\right),
\end{aligned} \tag{6.6}$$

where the first equality follows from [19, p. 385, Eq. (2)], and the second one from the Kummer transformation (1.26). Since i, j are nonnegative integers, one of the numbers $\frac{i+j}{2}$ and $\frac{i+j+1}{2}$ is a nonnegative integer and equals $[\frac{i+j+1}{2}]$; so,

$${}_2F_1\left(-\frac{i+j}{2}, -\frac{i+j+1}{2}; \frac{d}{2}; -\sigma^2\right) = \sum_{k=0}^{[\frac{i+j+1}{2}]} \frac{(-\frac{i+j}{2})_k(-\frac{i+j+1}{2})_k}{(\frac{d}{2})_k} \frac{(-1)^k\sigma^{2k}}{k!}. \tag{6.7}$$

Now, setting $\sigma := 2\sqrt{u}/(\mu+\nu)$ we substitute (6.7) into (6.6) and then put the result into (6.5); the conclusion is

$$G_{std}(\mu, \nu; u) = \frac{\pi}{\Gamma(d/2)\Gamma(s)\Gamma(t)} \sum_{i=0}^{s-\frac{d}{2}-\frac{1}{2}} \sum_{j=0}^{t-\frac{d}{2}-\frac{1}{2}} \sum_{k=0}^{[\frac{i+j+1}{2}]} \times \frac{(-1)^k(i+j+d-1)!(2s-i-d-1)!(2t-j-d-1)!(-\frac{i+j}{2})_k(-\frac{i+j+1}{2})_k}{2^{2s+2t-i-j-2k-d/2-3}i!j!k!(s-i-\frac{d}{2}-\frac{1}{2})!(t-j-\frac{d}{2}-\frac{1}{2})!(\frac{d}{2})_k} \times \frac{\mu^i\nu^j(\mu+\nu)^{i+j-2k+1}u^k}{((\mu+\nu)^2+4u)^{i+j+d/2+1/2}}. \tag{6.8}$$

The result (6.8) has the form

$$G_{std}(\mu, \nu; u) = \frac{\pi}{\Gamma(d/2)\Gamma(s)\Gamma(t)} \sum_{(ijk)\in I_{std}} G_{stijkd} \frac{\mu^i \nu^j (\mu+\nu)^{i+j-2k+1}(4u)^k}{((\mu+\nu)^2+4u)^{i+j+d/2+1/2}}, \tag{6.9}$$

where I_{std} and G_{stijkd} are as in Eqs. (2.36) and (2.37). The next step is to insert this result into Eq. (2.32) for $\|g_{\mu sd} g_{\nu td}\|^2_\ell$; this contains the integral over u of the expression

$$\begin{aligned}(1+4u)^\ell G^2_{std}(\mu, \nu; u) &= \frac{\pi^2}{\Gamma^2(d/2)\Gamma^2(s)\Gamma^2(t)} \sum_{h=0}^{\ell} \binom{\ell}{h} (4u)^h \\ &\quad \times \sum_{(ijk)\in I_{std}} \sum_{(i'j'k')\in I_{std}} G_{stijkd} G_{sti'j'k'd} \\ &\quad \times \frac{\mu^{i+i'} \nu^{j+j'} (\mu+\nu)^{i+i'+j+j'-2k-2k'+2} (4u)^{k+k'}}{((\mu+\nu)^2+4u)^{i+i'+j+j'+d+1}};\end{aligned} \tag{6.10}$$

we substitute this in (2.32) and integrate over u, taking into account that

$$\int_0^{+\infty} du\, \frac{(4u)^a}{(\xi+4u)^b} = \frac{\Gamma(a+1)\Gamma(b-a-1)}{4\Gamma(b)\xi^{b-a-1}}. \tag{6.11}$$

The conclusion is Eq. (2.35) for $\|g_{\mu sd} g_{\nu td}\|^2_\ell$.

7. Proof of Proposition 2.7

Throughout the section we make the assumptions of Eq. (2.46):

$$0 \leqslant \ell \leqslant n, \quad n > \frac{d}{2}.$$

Lemma 7.1. *One has*

$$K_{\ell\ell nd} \geqslant \frac{|g(0)|}{\|g\|_n} \tag{7.1}$$

for each nonzero $g \in H^n(\mathbf{R}^d, \mathbf{C})$. (*Note that* $g(0)$ *makes sense, by the well-known imbedding* $H^n(\mathbf{R}^d, \mathbf{C}) \subset C(\mathbf{R}^d, \mathbf{C})$.)

Proof. Let us present the idea heuristically. We fix a nonzero $g \in H^n(\mathbf{R}^d, \mathbf{C})$, and write the inequality

$$K_{\ell\ell nd} \geqslant \frac{\|f_\epsilon g\|_\ell}{\|f_\epsilon\|_\ell \|g\|_n} \tag{7.2}$$

where $(f_\epsilon)_{\epsilon>0}$ is a family of approximants of the Dirac δ distribution on $\mathbf{R}^d$: $f_\epsilon \to \delta$ as $\epsilon \to 0^+$. Then, for $\epsilon \to 0^+$, $f_\epsilon g \sim g(0) f_\epsilon$ and

$$\|f_\epsilon g\|_\ell \sim |g(0)| \|f_\epsilon\|_\ell; \tag{7.3}$$

so, in this limit, the inequality (7.2) gives the thesis (7.1). For a rigorization of this argument, see the proof of Lemma 7.1 in [14] (which contains a statement very similar to the present one). □

Proof of Proposition 2.7. From the previous lemma,

$$K_{\ell\ell nd} \geqslant \sup_{g \in H^n(\mathbf{R}^d, \mathbf{C}) \setminus \{0\}} \frac{|g(0)|}{\|g\|_n}; \tag{7.4}$$

as shown in [13] the above sup equals $S_{\infty nd}$, and is attained for $g = g_{nd}$ as in Eqs. (1.45) and (1.46). □

8. Proofs of Propositions 4.3 and 4.7

Each one of the two proofs will be preceded by a lemma about the asymptotics of a Laplace integral; we use this expression to indicate an integral of the form

$$L(\lambda) := \int_0^b dt\, \theta(t) e^{-\lambda\varphi(t)} \quad \big(b \in [0, +\infty),\ \lambda \in (\lambda_0, +\infty)\big) \tag{8.1}$$

where $\theta \in C((0, b), \mathbf{R})$, $\varphi \in C([0, b), \mathbf{R}) \cap C^1((0, b), \mathbf{R})$ are such that $\int_0^b dt |\theta(t)| e^{-\lambda\varphi(t)} < +\infty$ for all λ as above, and $\varphi'(t) > 0$ for $t \in (0, b)$ (the prime meaning d/dt). The following implication is well known (see e.g. [17]):

$$\begin{aligned} &\frac{\theta(t)}{\varphi'(t)} = P\big(\varphi(t) - \varphi(0)\big)^{\alpha-1}\big[1 + O\big(\varphi(t) - \varphi(0)\big)\big] \quad \text{for } t \to 0^+ \quad \big(P \in \mathbf{R},\ \alpha \in (0, +\infty)\big) \\ &\quad \Rightarrow \quad L(\lambda) = P e^{-\lambda\varphi(0)} \frac{\Gamma(\alpha)}{\lambda^\alpha}\left[1 + O\left(\frac{1}{\lambda}\right)\right] \quad \text{for } \lambda \to +\infty. \end{aligned} \tag{8.2}$$

Lemma 8.1. *Let*

$$L_\delta(\lambda) := \int_0^1 dt\, \frac{(1-t)^\lambda}{\sqrt{t}(3+t)^{3\lambda+\delta}} \quad \textit{for } \delta \in \mathbf{R},\ \lambda \in (0, +\infty). \tag{8.3}$$

Then, for each $\delta \in \mathbf{R}$,

$$L_\delta(\lambda) = \frac{1 + O(1/\lambda)}{3^{3\lambda+\delta}} \sqrt{\frac{\pi}{2\lambda}} \quad \textit{for } \lambda \to +\infty. \tag{8.4}$$

Proof. We have $L_\delta(\lambda) = \int_0^1 dt\, \theta_\delta(t) e^{-\lambda\varphi(t)}$, where

$$\theta_\delta(t) := \frac{1}{\sqrt{t}(3+t)^\delta}, \qquad \varphi(t) := 3\log(3+t) - \log(1-t). \tag{8.5}$$

It is easily checked that

$$\varphi'(t) = \frac{2(3-t)}{(1-t)(3+t)} > 0 \quad \text{for } t \in [0,1),$$
$$\varphi(0) = 3\log 3, \qquad \varphi(t) - \varphi(0) = 2t + O\big(t^2\big) \quad \text{for } t \to 0^+,$$
$$\frac{\theta_\delta(t)}{\varphi'(t)} = \frac{(\varphi(t)-\varphi(0))^{-1/2}}{\sqrt{2}\,3^\delta}\big[1 + O\big(\varphi(t)-\varphi(0)\big)\big] \quad \text{for } t \to 0^+; \tag{8.6}$$

so, application of (8.2) yields the thesis (8.4). □

Proof of Proposition 4.3. As usually, we consider any fixed space dimension $d \in \mathbf{N}_0$. We must prove condition $\mathfrak{S}_{22d}$, i.e.,

$$\sup_{u\in[0,+\infty)} \Sigma_{22d\ell}(u) = 1 + O(1/\ell) \quad \text{for } \ell \to +\infty. \tag{8.7}$$

Due to Eqs. (4.2) and (4.5), for each $u \geqslant 0$ we have

$$\Sigma_{22d\ell}(u) = (1+4u)^\ell\, {}_3F_2\left(4\ell - \frac{d}{2}, 2\ell, 2\ell; 2\ell, 2\ell + \frac{1}{2}; -u\right)$$
$$= (1+4u)^\ell\, {}_2F_1\left(4\ell - \frac{d}{2}, 2\ell; 2\ell + \frac{1}{2}; -u\right) \quad \text{for } u \geqslant 0,\ \ell > d/6 \tag{8.8}$$

(the last equality depends on Eq. (1.25)). Now, using for ${}_2F_1$ the integral representation (1.27) we get

$$\Sigma_{22d\ell}(u) = \frac{\Gamma(2\ell+1/2)}{\sqrt{\pi}\,\Gamma(2\ell)}\int_0^1 ds\, \frac{s^{2\ell-1}}{\sqrt{1-s}}\, W_{sd\ell}(u), \quad W_{sd\ell}(u) := \frac{(1+4u)^\ell}{(1+su)^{4\ell-d/2}}; \tag{8.9}$$

of course, this implies

$$\sup_{u\in[0,+\infty)} \Sigma_{22d\ell}(u) \leqslant \frac{\Gamma(2\ell+1/2)}{\sqrt{\pi}\,\Gamma(2\ell)}\int_0^1 ds\, \frac{s^{2\ell-1}}{\sqrt{1-s}}\Big(\sup_{u\in[0,+\infty)} W_{sd\ell}(u)\Big). \tag{8.10}$$

For all $\ell > d/6$ and $s \in (0,1)$, the function $W_{sd\ell} : [0,+\infty) \to (0,+\infty)$ attains its maximum at the point

$$u_{sd\ell} := \frac{1 - (1 - \frac{d}{8\ell})s}{3(1 - \frac{d}{6\ell})s}, \tag{8.11}$$

and so

$$\sup_{u\in[0,+\infty)} W_{sd\ell}(u) = W_{sd\ell}(u_{sd\ell}) = \frac{(\frac{3}{4})^{3\ell-d/2}(1-\frac{d}{6\ell})^{3\ell-d/2}}{(1-\frac{d}{8\ell})^{4\ell-d/2} s^\ell (1-\frac{s}{4})^{3\ell-d/2}}. \tag{8.12}$$

Inserting this result into Eq. (8.10) we get

$$\sup_{u\in[0,+\infty)} \Sigma_{22d\ell}(u) \leqslant U_{d\ell},$$

$$U_{d\ell} := \frac{(\frac{3}{4})^{3\ell-d/2}(1-\frac{d}{6\ell})^{3\ell-d/2}}{(1-\frac{d}{8\ell})^{4\ell-d/2}} \frac{\Gamma(2\ell+1/2)}{\sqrt{\pi}\,\Gamma(2\ell)} \int_0^1 ds \, \frac{s^{\ell-1}}{\sqrt{1-s}\,(1-\frac{s}{4})^{3\ell-d/2}}; \tag{8.13}$$

now, with a change of variable $s = 1-t$ in the integral and a comparison with Eq. (8.3), we find that

$$U_{d\ell} = \frac{3^{3\ell-d/2}(1-\frac{d}{6\ell})^{3\ell-d/2}}{(1-\frac{d}{8\ell})^{4\ell-d/2}} \frac{\Gamma(2\ell+1/2)}{\sqrt{\pi}\,\Gamma(2\ell)} L_{3-d/2}(\ell-1) \tag{8.14}$$

(the last factor indicates the Laplace integral $L_\delta(\lambda)$ of Eq. (8.3), with $\lambda = \ell-1$ and $\delta = 3-d/2$). Let us determine the behavior of $U_{d\ell}$ for $\ell \to +\infty$. To this purpose, we use the relations

$$\left(1-\frac{d}{6\ell}\right)^{3\ell-d/2} = e^{-d/2}\big[1+O(1/\ell)\big], \qquad \left(1-\frac{d}{8\ell}\right)^{4\ell-d/2} = e^{-d/2}\big[1+O(1/\ell)\big],$$

$$\frac{\Gamma(2\ell+1/2)}{\Gamma(2\ell)} = \sqrt{2\ell}\big[1+O(1/\ell)\big], \qquad L_{3-d/2}(\ell-1) = \frac{1+O(1/\ell)}{3^{3\ell-d/2}}\sqrt{\frac{\pi}{2\ell}}; \tag{8.15}$$

the first two are obvious, the third one follows from Eq. (1.15) and the fourth one comes from the asymptotics (8.4) of $L_\delta(\lambda)$. Inserting the relations (8.15) into (8.14), we get

$$U_{d\ell} = 1 + O(1/\ell). \tag{8.16}$$

Let us summarize Eqs. (8.13) and (8.16):

$$\sup_{u\in[0,+\infty)} \Sigma_{22d\ell}(u) \leqslant U_{d\ell} = 1 + O(1/\ell) \quad \text{for } \ell \to +\infty; \tag{8.17}$$

obviously enough, it is also

$$\sup_{u\in[0,+\infty)} \Sigma_{22d\ell}(u) \geqslant \Sigma_{22d\ell}(0) = 1 \tag{8.18}$$

and Eqs. (8.17) and (8.18) give the thesis (8.7). □

Remark 8.2. Using Eq. (8.8) with the known relation $(d/dw)|_{w=0}\,{}_2F_1(a,b,c,w) = ab/c$, one easily finds that

$$\left.\frac{d}{du}\right|_{u=0} \Sigma_{22d\ell}(u) = \frac{2(d+2)\ell}{4\ell+1} > 0. \tag{8.19}$$

So, the function $\Sigma_{22d\ell} : [0,+\infty) \to (0,+\infty)$ is strictly increasing in a neighborhood of $u=0$; we also remark that $(d/du)|_{u=0}\Sigma_{22d\ell}(u) \to d/2+1$ for $\ell \to +\infty$. Even though $u=0$ is not a maximum point, the $\ell \to +\infty$ asymptotics $\sup_{u\geqslant 0} \Sigma_{22d\ell}(u) = 1 + O(1/\ell) = \Sigma_{22d\ell}(0) + O(1/\ell)$ suggests that, for large ℓ, the sup of $\Sigma_{22d\ell}$ could be obtained at a point $O(1/\ell)$. We have found numerical evidence for this: $\Sigma_{22d\ell}$ seems to have a unique maximum point $u_{22d\ell}$, such that $u_{22d\ell} = O(1/\ell)$ for $\ell \to +\infty$.

Lemma 8.3. *Let $f_{\sigma d}(x) := e^{-\sigma|x|^2/2}$ for $x \in \mathbf{R}^d$ and $\sigma > 0$, as in* (4.12)*; furthermore, fix*

$$a \in (0, +\infty), \quad \zeta \in (0, 1/a). \tag{8.20}$$

Then, with $\| \, \|_{a\ell}$ indicating the $H^{a\ell}$ norm,

$$\|f_{\zeta/\ell,d}\|_{a\ell} = \left[\frac{\pi\ell}{\zeta(1-a\zeta)}\right]^{d/4}\left[1 + O\left(\frac{1}{\ell}\right)\right] \quad \text{for } \ell \to +\infty. \tag{8.21}$$

Proof. Eq. (2.42) gives

$$\|f_{\zeta/\ell,d}\|_{a\ell}^2 = \frac{2\pi^{d/2}\ell^d}{\Gamma(d/2)\zeta^d}\int_0^{+\infty} d\rho\, \rho^{d-1}\left(1+\rho^2\right)^{a\ell} e^{-\frac{\ell\rho^2}{\zeta}}; \tag{8.22}$$

with a change of variable $\rho = \sqrt{\zeta t}$, we get

$$\|f_{\zeta/\ell,d}\|_{a\ell}^2 = \frac{\pi^{d/2}\ell^d}{\Gamma(d/2)\zeta^{d/2}} L_{a\zeta d}(\ell), \quad L_{a\zeta d}(\ell) := \int_0^{+\infty} dt\, t^{d/2-1}(1+\zeta t)^{a\ell} e^{-\ell t}. \tag{8.23}$$

We note that

$$L_{a\zeta d}(\ell) = \int_0^{+\infty} dt\, \vartheta_d(t) e^{-\ell\varphi_{a\zeta}(t)},$$
$$\vartheta_d(t) := t^{d/2-1}, \quad \varphi_{a\zeta}(t) := t - a\log(1+\zeta t); \tag{8.24}$$

this indicates that $L_{a\zeta}(\ell)$ is a Laplace integral in the parameter ℓ, in the sense reviewed at the beginning of the section. One easily checks that

$$\varphi'_{a\zeta}(t) = \frac{1 - a\zeta + \zeta t}{1+\zeta t} > 0 \quad \text{for } t \in [0, +\infty),$$
$$\varphi_{a\zeta}(0) = 0, \qquad \varphi_{a\zeta}(t) = (1-a\zeta)t + O\left(t^2\right) \quad \text{for } t \to 0^+,$$
$$\frac{\vartheta_d(t)}{\varphi'_{a\zeta}(t)} = \frac{\varphi_{a\zeta}(t)^{d/2-1}}{(1-a\zeta)^{d/2}}\left[1 + O\left(\varphi_{a\zeta}(t)\right)\right] \quad \text{for } t \to 0^+; \tag{8.25}$$

from here and (8.2), we get

$$L_{a\zeta d}(\ell) = \frac{\Gamma(d/2)}{(1-a\zeta)^{d/2}\ell^{d/2}}\left[1 + O\left(\frac{1}{\ell}\right)\right] \quad \text{for } \ell \to +\infty. \tag{8.26}$$

Inserting (8.26) into (8.23), and taking the square root, we get the thesis (8.21). □

Proof of Proposition 4.7. Let $1 \leqslant b \leqslant c$ and $\xi \in (0, 1/b)$, $\eta \in (0, 1/c)$ with $\xi + \eta < 1$; we must derive the $\ell \to +\infty$ asymptotics (4.14), i.e.,

$$\frac{\|f_{\xi/\ell+\eta/\ell,d}\|_\ell}{\|f_{\xi/\ell,d}\|_{b\ell}\|f_{\eta/\ell,d}\|_{c\ell}} = \frac{1 + O(1/\ell)}{[D_{bc}(\xi,\eta)\pi\ell]^{d/4}}, \quad D_{bc}(\xi,\eta) := \frac{(1-\xi-\eta)(\xi+\eta)}{\xi\eta(1-b\xi)(1-c\eta)}. \tag{8.27}$$

The thesis follows using Eq. (8.21) with $(a, \zeta) = (1, \xi + \eta)$, or (b, ξ), or (c, η) (in each of the three cases, the assumptions on ξ, η ensure conditions (8.20) to be fulfilled). $\square$

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Appendix A. Derivation of Eq. (1.32)

Let us consider the integral

$$I_{\mu\nu\delta}(h) := \int_0^{+\infty} dr\, r^{\mu+\nu+\delta+1} J_\delta(hr) K_\mu(r) K_\nu(r); \tag{A.1}$$

with this notation, Eq. (1.32) reads

$$\begin{aligned} I_{\mu\nu\delta}(h) &= 2^{\mu+\nu+\delta-1} \frac{\Gamma(\mu+\delta+1)\Gamma(\nu+\delta+1)\Gamma(\mu+\nu+\delta+1)}{\Gamma(\mu+\nu+2\delta+2)} h^\delta \\ &\quad \times {}_3F_2\left(\mu+\delta+1, \nu+\delta+1, \mu+\nu+\delta+1; \frac{\mu+\nu}{2}+\delta+1, \frac{\mu+\nu}{2}+\delta+\frac{3}{2}; -\frac{h^2}{4}\right) \\ &\quad \text{for } h, \mu, \nu, \delta \in \mathbf{R},\ h > 0,\ \delta, \mu+\delta, \nu+\delta, \mu+\nu+\delta > -1. \end{aligned} \tag{A.2}$$

In the sequel we prove this identity, after checking preliminarily that the integral in the right-hand side converges under the above conditions for h, μ, ν, δ.

Convergence of the integral follows immediately from the relations $J_\xi(w) = O(w^\xi)$, $K_\eta(w) = O(w^{-|\eta|})$ for $\xi > -1$, $\eta \in \mathbf{R}$, $w \to 0^+$ and $J_\xi(w) = O(1/\sqrt{w})$, $K_\eta(w) = e^{-w} O(1/\sqrt{w})$ for $\xi, \eta \in \mathbf{R}$, $w \to +\infty$ (see [19, Chapters III and VII]); these ensure integrability of the function of r in $I_{\mu\nu\delta}(h)$, both near zero and near $+\infty$. To derive the equality (A.2), we start from the familiar series expansion (see again [19, Chapter III])

$$J_\delta(w) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\delta+1+k)} \left(\frac{w}{2}\right)^{\delta+2k}, \tag{A.3}$$

to be applied with $w = hr$; inserting this into Eq. (A.1), we get

$$I_{\mu\nu\delta}(h) = \left(\frac{h}{2}\right)^\delta \sum_{k=0}^{+\infty} \frac{1}{k!\Gamma(\delta+1+k)} \left(\frac{-h^2}{4}\right)^k \int_0^{+\infty} dr\, r^{2\delta+\mu+\nu+1+2k} K_\mu(r) K_\nu(r). \tag{A.4}$$

On the other hand,

$$\begin{aligned} &\int_0^{+\infty} dr\, r^{\alpha-1} K_\mu(r) K_\nu(r) \\ &\quad = \frac{2^{\alpha-3}}{\Gamma(\alpha)} \Gamma\left(\frac{\alpha-\mu-\nu}{2}\right) \Gamma\left(\frac{\alpha+\mu-\nu}{2}\right) \Gamma\left(\frac{\alpha-\mu+\nu}{2}\right) \Gamma\left(\frac{\alpha+\mu+\nu}{2}\right) \end{aligned} \tag{A.5}$$

if the arguments of all the above Gamma functions are positive (this is a special case of an identity in [7, see Eq. (6.576.4), p. 693]). We can use Eq. (A.5) to compute the integrals in (A.4), the conclusion being

$$I_{\mu\nu\delta}(h) = 2^{\mu+\nu+\delta-1} h^{\delta} \times \sum_{k=0}^{+\infty} \frac{1}{k!} \frac{\Gamma(\mu+\delta+1+k)\Gamma(\nu+\delta+1+k)\Gamma(\mu+\nu+\delta+1+k)}{\Gamma(\mu+\nu+2\delta+2+2k)} \left(-h^2\right)^k. \tag{A.6}$$

Now, we introduce the relations

$$\Gamma(\alpha+k) = (\alpha)_k \Gamma(\alpha), \qquad \Gamma(2\alpha+2k) = 4^k (\alpha)_k \left(\alpha+\frac{1}{2}\right)_k \Gamma(2\alpha) \quad \text{for } k \in \mathbf{N} \tag{A.7}$$

(the first appearing in Eq. (1.12), the second following from the first and from the elementary identity $(2\alpha)_{2k} = 4^k (\alpha)_k (\alpha+1/2)_k$). In this way we get

$$I_{\mu\nu\delta}(h) = 2^{\mu+\nu+\delta-1} \frac{\Gamma(\mu+\delta+1)\Gamma(\nu+\delta+1)\Gamma(\mu+\nu+\delta+1)}{\Gamma(\mu+\nu+2\delta+2)} h^{\delta} \times \sum_{k=0}^{+\infty} \frac{1}{k!} \frac{(\mu+\delta+1)_k(\nu+\delta+1)_k(\mu+\nu++\delta+1)_k}{(\frac{\mu+\nu}{2}+\delta+1)_k(\frac{\mu+\nu}{2}+\delta+\frac{3}{2})_k} \left(-\frac{h^2}{4}\right)^k. \tag{A.8}$$

According to Eq. (1.18), the above series equals

$${}_3F_2\left(\mu+\delta+1, \nu+\delta+1, \mu+\nu+\delta+1; \frac{\mu+\nu}{2}+\delta+1, \frac{\mu+\nu}{2}+\delta+\frac{3}{2}; -\frac{h^2}{4}\right),$$

so Eq. (A.2) is proved. (Final remark: in fact, the previous considerations give the thesis (A.2) for $h^2/4 < 1$, i.e. $h \in (0,2)$, since the series expansion (1.18) for ${}_3F_2$ has a convergence radius 1. However, after proving the thesis for $h \in (0,2)$ one can extend it to all $h \in (0,+\infty)$ by a standard application of the analytic continuation principle.)

Appendix B. Calculation of the upper and lower bounds $K^{\pm}_{\ell mnd}$ in Table 1: Some examples

Computation of K^{+}_{0121}. (i) We first determine the $\mathscr{S}$-function upper bound. Eqs. (2.4)–(2.14) give

$$\mathscr{S}_{0121}(u) = \frac{3+u}{16(1+u)^2} \quad \text{for } u \in [0,+\infty); \tag{B.1}$$

the above function is easily studied by analytical means, the conclusion being

$$\sup_{u\in[0,+\infty)} \mathscr{S}_{0121}(u) = \mathscr{S}_{0121}(0) = \frac{3}{16}. \tag{B.2}$$

So,

$$\sqrt{\sup_{u\in[0,+\infty)} \mathscr{S}_{0121}(u)} = \sqrt{\frac{3}{16}} < 0.434 := K^{\mathscr{S}}_{0121}. \tag{B.3}$$

(ii) Let us build the Hölder upper bound (3.2); in this case, Eqs. (2.16), (2.23) and (2.25) give $R_{11} = R_{21} = R_{121} = [2, +\infty]$, so we must evaluate $\inf_{p\in[2,+\infty]} S_{p11} S_{p^*21}$, the factors S_{p11}, S_{p^*21} being given by Eqs. (2.18)–(2.20). As found numerically, the inf is attained for p close to 3.21, and

$$\inf_{p\in[2,+\infty]} S_{p11} S_{p^*21} \leqslant 0.383 := K^{\mathscr{H}}_{0121}. \tag{B.4}$$

(iii) The Hölder bound $K^{\mathscr{H}}_{0121}$ is better than the $\mathscr{S}$-function bound $K^{\mathscr{S}}_{0121}$, so we take

$$K^{+}_{0121} := K^{\mathscr{H}}_{0121} = 0.383; \tag{B.5}$$

this is the value reported in the table.

Computation of K^{-}_{0121}. (i) We first consider the Bessel lower bound (3.4) with $s = 1$, $t = 2$. In this case, Eqs. (2.30) and (2.35) give

$$\|g_{\mu 11}\|_1^2 = \frac{\pi}{2}\frac{1+\mu^2}{\mu}, \qquad \|g_{\nu 21}\|_2^2 = \frac{\pi}{16}\frac{5+2\nu^2+\nu^4}{\nu}, \tag{B.6}$$

$$\|g_{\mu 11} g_{\nu 21}\|_0^2 = \frac{\pi^2}{32}\frac{2\mu^2+6\mu\nu+5\nu^2}{(\mu+\nu)^3}; \tag{B.7}$$

from here one computes, according to Eq. (2.38), the function

$$\mathscr{K}^{B_{12}}_{0121}(\mu,\nu) := \frac{\|g_{\mu 11} g_{\nu 21}\|_0}{\|g_{\mu 11}\|_1 \|g_{\nu 21}\|_2} \quad \bigl(\mu,\nu \in (0,+\infty)\bigr). \tag{B.8}$$

It is found numerically that the above function attains its sup for (μ,ν) close to $(0.499, 0.784)$, and that

$$\sup_{\mu,\nu>0} \mathscr{K}^{B_{12}}_{0121}(\mu,\nu) \geqslant 0.951 K^{+}_{0121} := K^{B_{12}}_{0121}. \tag{B.9}$$

(ii) We pass to the Fourier lower bound (3.5). In this case, from Eq. (2.43) one gets

$$\|f_{h\kappa 1}\|_0^2 = \sqrt{\frac{\pi}{\kappa}}, \qquad \|f_{p\sigma 1}\|_1^2 = \frac{\sqrt{\pi}}{2}\frac{2+2p^2+\sigma}{\sqrt{\sigma}},$$
$$\|f_{q\tau 1}\|_2^2 = \frac{\sqrt{\pi}}{4}\frac{4+8q^2+4q^4+4\tau+12q^2\tau+3\tau^2}{\sqrt{\tau}} \tag{B.10}$$

for $h, p, q \in [0,+\infty)$ and $\kappa, \sigma, \tau \in (0,+\infty)$; from here, one computes the function

$$\mathscr{K}^{F}_{0121}(p,q,\sigma,\tau) := \frac{\|f_{p+q,\sigma+\tau,1}\|_0}{\|f_{p\sigma 1}\|_1 \|f_{q\tau 1}\|_2} \tag{B.11}$$

$(p, q \in [0,+\infty),\ \sigma,\tau \in (0,+\infty))$. A numerical investigation seems to indicate that the sup of this function is attained for (p,q,σ,τ) close to $(0, 0, 0.472, 0.291)$; in any case, using the value at this point as a lower approximant for the sup we get

$$\sup_{p,q\geqslant 0,\,\sigma,\tau>0} \mathscr{K}^{F}_{0121}(p,q,\sigma,\tau) \geqslant 0.987 K^{+}_{0121} := K^{F}_{0121}. \tag{B.12}$$

(iii) The Fourier lower bound K^{F}_{0121} is better than the Bessel lower bound $K^{B_{12}}_{0121}$; in conclusion we take

$$K^{-}_{0121} := K^{F}_{0121} = 0.987 K^{+}_{0121}, \tag{B.13}$$

as indicated in the table. The symbol (F) appearing in the table recalls that the lower bound K^{-}_{0121} is of the Fourier type.

Computation of K^{+}_{4561}. We use for this the $\mathscr{S}$-function upper bound. Eqs. (2.4)–(2.14) give

$$\mathscr{S}_{4561}(u) = (1+4u)^4 \frac{46189 + 20995u + 9690u^2 + 3230u^3 + 665u^4 + 63u^5}{524288(1+u)^{10}} \tag{B.14}$$

for $u \in [0, +\infty)$. It is found numerically that the above function attains its sup close to $u = 0.315$, and that

$$\sqrt{\sup_{u \in [0,+\infty)} \mathscr{S}_{4561}(u)} \leqslant 0.417 := K^{+}_{4561}; \tag{B.15}$$

this upper bound is reported in the table.

Computation of K^{-}_{4561}. (i) We first consider the Bessel lower bound (3.4) with $s = 5$, $t = 6$. Eq. (2.30) gives

$$\|g_{\mu 51}\|_5^2 = \frac{5\pi}{65536\mu}\bigl(2431 + 715\mu^2 + 286\mu^4 + 110\mu^6 + 35\mu^8 + 7\mu^{10}\bigr),$$
$$\|g_{\nu 61}\|_6^2 = \frac{3\pi}{524288\nu}\bigl(29393 + 8398\nu^2 + 3315\nu^4 + 1300\nu^6 + 455\nu^8 + 126\nu^{10} + 21\nu^{12}\bigr) \tag{B.16}$$

for $\mu, \nu \in (0, +\infty)$. Eq. (2.35) gives

$$\|g_{\mu 51} g_{\nu 61}\|_4^2 = \frac{\pi^2}{34359738368(\mu+\nu)^{19}} P(\mu, \nu) \tag{B.17}$$

where $P(\mu, \nu)$ is a polynomial of the form:

$$P(\mu, \nu) = \sum_{i,j \in \mathbf{N},\, 18 \leqslant i+j \leqslant 26} P_{ij} \mu^i \nu^j, \quad P_{ij} \in \mathbf{N} \text{ for all } i, j. \tag{B.18}$$

The full expression of this polynomial is easily computed with MATHEMATICA, but it is too long to be reported here; as examples we give only three coefficients, namely,

$$P_{18,0} = 192972780, \qquad P_{1,25} = 4236050, \qquad P_{0,26} = 222950. \tag{B.19}$$

The expressions of the above norms determine the function

$$\mathscr{K}^{B_{56}}_{4561}(\mu, \nu) := \frac{\|g_{\mu 51} g_{\nu 61}\|_4}{\|g_{\mu 51}\|_5 \|g_{\nu 61}\|_6} \quad \bigl(\mu, \nu \in (0, +\infty)\bigr). \tag{B.20}$$

It is found numerically that the above function attains its sup for (μ, ν) close to $(1.19, 1.14)$, and that

$$\sup_{\mu,\nu>0} \mathscr{K}^{B_{56}}_{4561}(\mu,\nu) \geqslant 0.823 K^{+}_{4561} := \mathscr{K}^{B_{56}}_{4561}. \tag{B.21}$$

(ii) We pass to the Fourier lower bound (3.5). From Eq. (2.43) one gets

$$\begin{aligned}
\|f_{h\kappa 1}\|_4^2 = {} & \frac{1}{16}\sqrt{\frac{\pi}{\kappa}}\big(16 + 64h^2 + 96h^4 + 64h^6 + 16h^8 + 32\kappa + 288h^2\kappa \\
& + 480h^4\kappa + 224h^6\kappa + 72\kappa^2 + 720h^2\kappa^2 + 840h^4\kappa^2 + 120\kappa^3 + 840h^2\kappa^3 + 105\kappa^4\big), \\
\|f_{p\sigma 1}\|_5^2 = {} & \frac{1}{32}\sqrt{\frac{\pi}{\sigma}}\big(32 + 160p^2 + 320p^4 + 320p^6 + 160p^8 + 32p^{10} + 80\sigma \\
& + 960p^2\sigma + 2400p^4\sigma + 2240p^6\sigma + 720p^8\sigma + 240\sigma^2 + 3600p^2\sigma^2 + 8400p^4\sigma^2 \\
& + 5040p^6\sigma^2 + 600\sigma^3 + 8400p^2\sigma^3 + 12600p^4\sigma^3 + 1050\sigma^4 + 9450p^2\sigma^4 + 945\sigma^5\big), \\
\|f_{q\tau 1}\|_6^2 = {} & \frac{1}{64}\sqrt{\frac{\pi}{\tau}}\big(64 + 384q^2 + 960q^4 + 1280q^6 + 960q^8 + 384q^{10} + 64q^{12} + 192\tau \\
& + 2880q^2\tau + 9600q^4\tau + 13440q^6\tau + 8640q^8\tau + 2112q^{10}\tau + 720\tau^2 + 14400q^2\tau^2 \\
& + 50400q^4\tau^2 + 60480q^6\tau^2 + 23760q^8\tau^2 + 2400\tau^3 + 50400q^2\tau^3 + 151200q^4\tau^3 \\
& + 110880q^6\tau^3 + 6300\tau^4 + 113400q^2\tau^4 + 207900q^4\tau^4 \\
& + 11340\tau^5 + 124740q^2\tau^5 + 10395\tau^6\big)
\end{aligned} \tag{B.22}$$

for $h, p, q \in [0, +\infty)$ and $\kappa, \sigma, \tau \in (0, +\infty)$; from here, one computes the function

$$\mathscr{K}^{F}_{4561}(p,q,\sigma,\tau) := \frac{\|f_{p+q,\sigma+\tau,1}\|_4}{\|f_{p\sigma 1}\|_5 \|f_{q\tau 1}\|_6} \tag{B.23}$$

($p, q \in [0, +\infty)$, $\sigma, \tau \in (0, +\infty)$). A numerical investigation seems to indicate that the sup of this function is attained for (p, q, σ, τ) close to $(0.288, 0.215, 0.147, 0.109)$; in any case, using the value at this point as a lower approximant for the sup we get

$$\sup_{p,q\geqslant 0,\sigma,\tau>0} \mathscr{K}^{F}_{4561}(p,q,\sigma,\tau) \geqslant 0.878 K^{+}_{4561} := K^{F}_{4561}. \tag{B.24}$$

(iii) The Fourier lower bound K^{F}_{4561} is better than the Bessel lower bound $K^{B_{56}}_{4561}$; in conclusion we take

$$K^{-}_{4561} := K^{F}_{4561} = 0.878 K^{+}_{4561}, \tag{B.25}$$

as indicated in the table. The symbol (F) appearing in the table recalls the type of the lower bound K^{-}_{4561}.

Computation of K^{+}_{1123}. We use for this the $\mathscr{S}$-function upper bound. Eqs. (2.4)–(2.14) give

$$\mathscr{S}_{1123}(u) = \frac{(1+4u)}{32\pi(1+u)} \tag{B.26}$$

for $u \in [0, +\infty)$. The above function attains its sup in the limit $u \to +\infty$, and

$$\sqrt{\sup_{u\in[0,+\infty)} \mathscr{S}_{1123}(u)} = \sqrt{\mathscr{S}_{1123}(+\infty)} = \frac{1}{2\sqrt{2\pi}} := K^{+}_{1123}. \tag{B.27}$$

This is the value reported in the table; from a numerical viewpoint, $K^{+}_{1123} = 0.1994\ldots$.

Computation of K^{-}_{1123}. We are discussing a case with $\ell = m$, so we have the S-constant lower bound (2.47); more precisely, this bound is (recalling Eq. (2.20))

$$S_{\infty 23} = \frac{1}{2\sqrt{2\pi}} := K^{-}_{1123}. \tag{B.28}$$

This lower bound equals K^{+}_{1123}; we can avoid calculating the Bessel and Fourier lower bounds, since they cannot be better. In the table we have indicated that $K^{-}_{1123}/K^{+}_{1123} = 1$, and we have used the symbol (S) to recall the type of the lower bound.

Of course, in this case we have the sharp constant:

$$K_{1123} = K^{\pm}_{1123}. \tag{B.29}$$

Computation of K^{+}_{2233}. Again, we use the $\mathscr{S}$-function bound. Eqs. (2.4)–(2.14) give

$$\mathscr{S}_{2233}(u) = \frac{(1+4u)^2(5+u)}{512\pi(1+u)^3} \tag{B.30}$$

for $u \in [0, +\infty)$. It is found that

$$\sqrt{\sup_{u\in[0,+\infty)} \mathscr{S}_{2233}(u)} = \sqrt{\mathscr{S}_{2233}\left(\frac{13}{5}\right)} = \frac{19}{288}\sqrt{\frac{19}{2\pi}} < 0.115 := K^{+}_{2233}; \tag{B.31}$$

this upper bound is reported in the table.

Computation of K^{-}_{2233}. (i) Let us compute the Bessel lower bound (3.4), with $s = 2$, $t = 3$. Eqs. (2.30) and (2.35) give

$$\|g_{\mu 23}\|_2^2 = \frac{\pi^2}{8\mu^3}\left(1 + 2\mu^2 + 5\mu^4\right), \qquad \|g_{\nu 33}\|_3^2 = \frac{\pi^2}{128\nu^3}\left(7 + 9\nu^2 + 9\nu^4 + 7\nu^6\right), \tag{B.32}$$

$$\begin{aligned}\|g_{\mu 23} g_{\nu 33}\|_2^2 = \frac{\pi^3}{1024(\mu+\nu)^5}\big(&\mu^2 + 2\mu^4 + 5\mu^6 + 5\mu\nu + 10\mu^3\nu + 25\mu^5\nu \\ &+ 7\nu^2 + 20\mu^2\nu^2 + 53\mu^4\nu^2 + 18\mu\nu^3 + 62\mu^3\nu^3 + 6\nu^4 \\ &+ 43\mu^2\nu^4 + 17\mu\nu^5 + 3\nu^6\big);\end{aligned} \tag{B.33}$$

from here one computes, according to Eq. (2.38), the function

$$\mathscr{K}^{B_{23}}_{2231}(\mu, \nu) := \frac{\|g_{\mu 23} g_{\nu 33}\|_2}{\|g_{\mu 23}\|_2 \|g_{\nu 33}\|_3} \quad \big(\mu, \nu \in (0, +\infty)\big). \tag{B.34}$$

It is found numerically that the above function attains its sup for (μ, ν) close to $(1.31, 1.04)$, and that

$$\sup_{\mu,\nu>0} \mathscr{K}^{B_{23}}_{2231}(\mu,\nu) \geqslant 0.916 K^{+}_{2231} := \mathscr{K}^{B_{23}}_{2231}. \tag{B.35}$$

(ii) Let us pass to the Fourier lower bound (3.5). From Eq. (2.43) one gets

$$\begin{aligned} \|f_{p\sigma 3}\|_2^2 &= \frac{1}{4}\left(\frac{\pi}{\sigma}\right)^{3/2}\left(4+8p^2+4p^4+12\sigma+20p^2\sigma+15\sigma^2\right), \\ \|f_{q\tau 3}\|_3^2 &= \frac{1}{8}\left(\frac{\pi}{\tau}\right)^{3/2}\left(8+24q^2+24q^4+8q^6+36\tau+120q^2\tau+84q^4\tau \right. \\ &\quad \left. +90\tau^2+210q^2\tau^2+105\tau^3\right), \end{aligned} \tag{B.36}$$

for $p, q \in [0, +\infty)$ and $\sigma, \tau \in (0, +\infty)$; from here, one computes the function

$$\mathscr{K}^{F}_{2233}(p,q,\sigma,\tau) := \frac{\|f_{p+q,\sigma+\tau,3}\|_2}{\|f_{p\sigma 3}\|_2 \|f_{q\tau 3}\|_3} \tag{B.37}$$

$(p, q \in [0, +\infty),\ \sigma, \tau \in (0, +\infty))$. A numerical investigation seems to indicate that the sup of this function is attained for (p, q, σ, τ) close to $(0.667,\ 0.114,\ 2.53,\ 0.430)$; in any case, using the value at this point as a lower approximant for the sup we get

$$\sup_{p,q\geqslant 0,\, \sigma,\tau>0} \mathscr{K}^{F}_{2233}(p,q,\sigma,\tau) \geqslant 0.809 K^{+}_{2233} := K^{F}_{2233}. \tag{B.38}$$

(iii) Since we are discussing a case with $\ell = m$, we have also the S-constant lower bound (2.47); this bound is (recalling Eq. (2.20))

$$S_{\infty 33} = \frac{1}{4\sqrt{2\pi}} = 0.8672\ldots\ K^{+}_{2233}. \tag{B.39}$$

(iv) The Bessel lower bound $K^{B_{23}}_{2233}$ is better than the S-constant and Fourier lower bounds $S_{\infty 33}$, K^{F}_{2233}; in conclusion we take

$$K^{-}_{2233} := K^{B_{23}}_{2233} = 0.916 K^{+}_{2233}, \tag{B.40}$$

as indicated in the table. The symbol (B_{23}) appearing in the table recalls the type of the lower bound.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1992.
[2] N. Aronszajn, K.T. Smith, Theory of Bessel potentials I, Ann. Inst. Fourier (Grenoble) 11 (1961) 385–475.
[3] E.W. Barnes, The asymptotic expansions of integral functions defined by generalized hypergeometric series, Proc. London Math. Soc. (2) 5 (1907) 59–116.
[4] S. Benzoni-Gavage, D. Serre, Multidimensional Hyperbolic Partial Differential Equations. First-Order Systems and Applications, Oxford Univ. Press, Oxford, 2007.
[5] S. Bochner, K. Chandrasekharan, Fourier Transforms, Princeton Univ. Press, Princeton, 1949.
[6] H. Cartan, Elementary Theory of Analytic Functions of One or Several Complex Variables, Dover, New York, 1995.
[7] I.S. Gradshteyn, I.M. Ryzhik, Tables of Integrals, Series, and Products, Academic Press, New York, 1980.
[8] D. Karp, S.M. Sitnik, Inequalities and monotonicity of ratios for generalized hypergeometric function, J. Approx. Theory 161 (1) (2009) 337–352, doi:10.1016/j.jat.2008.10.002.

[9] P.E. Lammert, Differentiability of Lieb functional in electronic density functional theory, Internat. J. Quantum Chem. 107 (2007) 1943–1953.
[10] E.H. Lieb, An L^p bound for the Riesz and Bessel potentials of orthonormal functions, J. Funct. Anal. 51 (1983) 159–165.
[11] Y.L. Luke, The Special Functions and Their Approximations, Academic Press, New York, 1969.
[12] V.G. Mazjia, Sobolev Spaces, Springer, Berlin, 1985.
[13] C. Morosi, L. Pizzocchero, On the constants for some Sobolev imbeddings, J. Inequal. Appl. 6 (2001) 665–679.
[14] C. Morosi, L. Pizzocchero, On the constants in some inequalities for the Sobolev norms and pointwise product, J. Inequal. Appl. 7 (2002) 421–452.
[15] C. Morosi, L. Pizzocchero, On approximate solutions of semilinear evolution equations, Rev. Math. Phys. 16 (2004) 383–420; C. Morosi, L. Pizzocchero, On approximate solutions of semilinear evolution equations II. Generalizations, and applications to Navier–Stokes equations, Rev. Math. Phys. 20 (2008) 625–706.
[16] C. Morosi, L. Pizzocchero, On the constants for multiplication in Sobolev spaces, Adv. in Appl. Math. 36 (2006) 319–363.
[17] F.W.J. Olver, Asymptotics and Special Functions, Academic Press, San Diego, CA, 1974.
[18] A.P. Prudnykov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, vol. 3: Additional Chapters, Nauka, Moscow, 1986.
[19] G.N. Watson, A Treatise on the Theory of Bessel Functions, reprint of the second (1944) edition, Cambridge Math. Lib., Cambridge Univ. Press, 1995.